Qualitative Analysis of
Nonlinear Elliptic Partial Differential Equations

Vicenţiu D. Rădulescu

Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods

Contemporary Mathematics and Its Applications, Volume 6

Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods

Vicenţiu D. Rădulescu

Hindawi Publishing Corporation
http://www.hindawi.com

Contemporary Mathematics and Its Applications
Series Editors: Ravi P. Agarwal and Donal O'Regan

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA Toll-Free)

ISBN 978-977-454-039-4

Dedication

Dedicated to My Mother, with Gratitude and Love

Contents

Introduction

If I have seen further it is by standing on the shoulders of giants.

Sir Isaac Newton (1642–1727),
Letter to Robert Hooke (1675)

Nonlinear analysis is one of the fields with the most spectacular development in the last decades. The impressive number of results in this area is also a consequence of various problems raised by mathematical physics, optimization, or economy. In the modeling of natural phenomena, a crucial role is played by the study of partial differential equations of elliptic type. They arise in every field of science. Consequently, the desire to understand the solutions of these equations has always a prominent place in the efforts of mathematicians. It has inspired such diverse fields as functional analysis, variational calculus, potential theory, algebraic topology, differential geometry, and so forth.

The aim of this book is to present an introduction to a sampling of ideas, phenomena, and methods from the subject of nonlinear elliptic equations. One of the main goals of this textbook is to provide the background which is necessary to initiate research work in applied nonlinear analysis. My purpose is to provide for the student a broad perspective in the subject, to illustrate the rich variety of phenomena encompassed by it, and to impart a working knowledge of the most important techniques of analysis of the solutions of the equations. The level of this book is aimed at beginning graduate students. Prerequisites include a truly advanced calculus course, basic knowledge on functional analysis and linear elliptic equations, as well as the necessary tools on Sobolev spaces.

In this book, we are concerned with some basic monotonicity, analytic, and variational methods which are directly related to the theory of nonlinear partial differential equations of elliptic type. The abstract theorems are applied both to single-valued and to multivalued boundary value problems. We develop in this book abstract results in the following three directions: the maximum principle for nonlinear elliptic equations, the implicit function theorem, and the critical point theory. In the first category, we are concerned with the method of lower and upper solutions, which is the basic monotonicity principle for proving existence results. The implicit function theorem is used in connection with bifurcation problems, which play a crucial role to explain various phenomena that have been discovered and described in natural sciences over the centuries. The implicit function theorem is one of the oldest paradigms in analysis and is a strong tool in the qualitative analysis of nonlinear problems. The critical point theory is a deep instrument in the calculus of variations and its roots go back to the papers of G. Birkhoff and M. Morse. We are concerned with nonsmooth variants of several celebrated results: the Ambrosetti-Rabinowitz mountain pass theorem, the Rabinowitz saddle point theorem, and the Ljusternik-Schnirelmann theory for periodic functionals. All these theories establish that the solutions of wide classes of nonlinear boundary value problems coincide with the critical points of a natural functional (called "energy") on an appropriate manifold. This is not surprising since many of the laws of mathematics and physics can be formulated in terms of "extremum principles." Energy methods are of special interest in those situations in which traditional arguments based on comparison principles have failed. Energy methods are well suited in the study of differential systems which include equations of different types frequently arising in the mathematical models of continuum mechanics. In some of these cases, even if the comparison principle holds, it may be extremely difficult to construct suitable sub- and supersolutions if, for instance, the equation under study contains transport terms and has either variable or unbounded coefficients in the right-hand side. The main idea of the energy methods consists in deriving and studying suitable inequality problems for various types of energy. In typical situations, these inequalities follow from the conservation and balance laws of continuum mechanics. In the simplest situations, the energy functions defined through a formal procedure coincide with the kinetic and potential energies.

In the first part of this textbook, we present the method of lower and upper solutions (sub- and supersolutions), which is one of the main tools in nonlinear analysis for finding solutions to many classes of boundary value problems. The proofs are simple and we give several examples to illustrate the theory. We also focus on the stability of solutions and on some basic existence and uniqueness results (Brezis-Oswald and Krasnoselski's theorems). We illustrate the Brezis-Oswald theorem with a sublinear elliptic equation with variable potential on the whole space and we establish sufficient conditions for the existence of a unique solution decaying to different values at infinity.

In Chapter 2, we continue with another elementary method for finding solutions, namely, the implicit function theorem. The main abstract result in this chapter is a celebrated bifurcation theorem whose roots go back to the paper by Keller and Cohen [119]. We are also concerned in this chapter with the bifurcation of entire classical solutions to the logistic equation in anisotropic media. Bifurcation problems of this type, as well as the associated evolution equations, are naturally related to certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [57] as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. Such equations appear in the study of cellular automata and interacting particle systems with self-organized criticality, and also describe the flow over an impermeable plate. We also mention Brusselator-type reactions, the combustion theory, dynamics of population, the Fitzhugh-Nagumo system, morphogenesis, superconductivity, superfluids, and so forth.

In Chapter 3, we give some basic results related to the Clarke generalized gradient of a locally Lipschitz functional. Then we develop a nonsmooth critical point theory which enables us to deduce the Brezis-Coron-Nirenberg theorem, the saddle point theorem of P. Rabinowitz, or the Ghoussoub-Preiss theorem. The motivation of this study is that some of the strongest tools for proving existence results in nonlinear PDEs are the mountain pass theorem of A. Ambrosetti and P. Rabinowitz and the Ljusternik-Schnirelmann theorem. These results apply when the solutions of the given problem are critical points of a suitable energy functional E, which is assumed to be of class C^1 and is defined on a real

Banach space X. A natural question is what happens if the energy functional, associated to a given problem in a natural way, fails to be differentiable. The results we give here are based on the notion of Clarke generalized gradient, which coincides with the usual one if E is either differentiable or convex. In the *classical smooth* framework, the Fréchet differential of a C^1-functional $f : X \to \mathbb{R}$ is a linear and continuous operator. For the case of locally Lipschitz maps, the property of linearity of the gradient does not remain valid. Thus, for fixed $x \in X$, the directional derivative $f^0(x; \cdot)$ is subadditive and positive homogeneous and its generalized gradient $\partial f(x)$ is a nonempty closed subset of the dual space.

In Chapter 4, the main results are two abstract Ljusternik-Schnirelmann-type theorems. The first one uses the notion of critical point for a pairing of operators which was introduced by Fucik et al. [82]. The second theorem is related to locally Lipschitz functionals which are periodic with respect to a discrete subgroup and it also uses the notion of Clarke subdifferential. In a related framework, we study a nonlinear eigenvalue problem arising in earthquake initiation and we are concerned with the effect of a small perturbation for problems with symmetry.

In the following part of this work, we give several applications of the abstract results which appear in the first chapters. Here we recall a classical result of Chang [41] and prove multivalued variants of the Brezis-Nirenberg problem, as well as a solution of the forced pendulum problem, which was studied in the smooth case by Mawhin and Willem [143]. We are also concerned with multivalued problems at resonance of Landesman-Lazer type. The methods we develop here enable us to study several classes of discontinuous problems and all these techniques are based on Clarke's generalized gradient theory. This tool is very useful in the study of critical periodic orbits of Hamiltonian systems (F. Clarke and I. Ekeland), the mathematical programming (J. B. Hirriart-Urruty), the duality theory (T. Rockafellar), optimal control (V. Barbu and F. Clarke), nonsmooth analysis (F. Clarke, M. Degiovanni, and A. Ioffe), hemivariational inequalities (P. D. Panagiotopoulos), and so forth. Other applications considered in the last chapter of this book concern the Hartman-Stampacchia theory for hemivariational inequalities and the study of standing waves of multivalued Schrödinger equations or systems.

Part of this textbook was taught as a one-semester Ph.D. course at the University of Craiova, Romania, during 2000 and 2008, and in the Spring of 2006 to the students of the *École Normale Supérieure* in Bucharest.

The book is addressed to a wide audience of scientists, including researchers in applied mathematics and related areas, to graduate or postgraduate students, and, in particular, to those interested in stationary nonlinear boundary value problems. It can serve as a reference book on the existence theory for multivalued boundary value problems of elliptic partial differential equations, as well as the textbook for graduate or postgraduate classes. The techniques which we discuss and describe in this textbook go far beyond all the equations we study. In particular, the methods we develop in the present work can be applied to nonlinear wave equations, Hamiltonian systems, hemivariational inequalities, as well as problems related to surfaces of prescribed mean curvature. With this book, the student and the advanced researcher too are introduced to a rich tapestry of nonlinear analysis as it interacts with other parts of mathematics.

Finally, I am happy to express my thanks to Bianca and Teodora for their constant help and support.

Acknowledgments

The quality of the presentation of this book was greatly improved, thanks to the lectures given in various universities: Paris VI, Amiens, Chambéry, Poitiers, Limoges, Nancy, Metz, Strasbourg, Montpellier, Orléans, La Rochelle, Mulhouse, Perpignan, Bruxelles, Louvain, Leyden, Delft, Oxford, Brighton, Berlin, Aachen, Stockholm, Uppsala, Rome, Milan, Pisa, Florence, Naples, Bologna, Torino, Perugia, Brescia, Trento, Padova, Basel, Madrid, Budapest, Bucharest, Hong Kong, Tunis, and so forth. Many of the ideas and problems developed in this book came from my Ph.D. and Habilitation theses supervisor, Professor Haim Brezis. I would like to thank him warmly not anly for his constant and enthusiastic support over the years, but also for showing me the fascinating path to the modern nonlinear analysis.

March 2008

Vicenţiu D. Rădulescu

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1764, 014700 Bucharest, Romania

Department of Mathematics, University of Craiova, 200585 Craiova, Romania

vicentiu.radulescu@math.cnrs.fr

vicentiu.radulescu@imar.ro

http://www.inf.ucv.ro/~radulescu/

1 Monotonicity methods for stationary equations

Nature and Nature's law lay hid in night: God said, "Let Newton be!," and all was light.

Alexander Pope (1688–1744),
Epitaph on Newton

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Albert Einstein (1879–1955)

1.1. Generalities about the maximum principle

Let Ω be a bounded open set in $\mathbb{R}^N$. The maximum principle asserts that if $u : \overline{\Omega} \to \mathbb{R}$ is a smooth function such that

$$\begin{aligned} -\Delta u \geq 0 \quad &\text{in } \Omega, \\ u \geq 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

then $u \geq 0$ in Ω. A stronger form of the maximum principle was deduced by Hopf [107] in 1952. The Hopf lemma asserts that in the situation above, the following alternative holds: either u is constant in Ω or u is positive and nonconstant in Ω and, in this case, if $u = 0$ at some smooth boundary point, then the exterior normal derivative $\partial u/\partial \nu$ at that point is negative. In particular, the strong Hopf maximum principle establishes that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is superharmonic in Ω, and $u = 0$ on $\partial\Omega$, then either $u \equiv 0$ in Ω or $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$ (assuming that $\partial\Omega$ has the interior sphere property at any point).

Over the years, many variations and new ways of applying the maximum principle have been discovered, in order to prove estimates or to derive qualitative properties of solutions. In [211], Stampacchia showed that the strong maximum principle still remains true if we replace the Laplace operator with a coercive operator. More precisely, let $a \in L^\infty(\Omega)$ be such that, for some $\alpha > 0$,

$$\int_\Omega \left(|\nabla u|^2 + a(x)u^2\right) dx \geq \alpha \|u\|^2_{H^1_0(\Omega)}, \tag{1.2}$$

for all $u \in H_0^1(\Omega)$. Stampacchia's maximum principle asserts that if

$$\begin{aligned} -\Delta u + a(x)u &\geq 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

then either $u \equiv 0$ in Ω or $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$.

In particular, the hypotheses of Stampacchia's maximum principle are fulfilled if $a(x) \equiv 1$. Vázquez observed in [221] that we can replace the linear coercive operator $-\Delta + I$ by much more general operators, subject to monotonicity assumptions. More precisely, let $f : [0, +\infty) \to \mathbb{R}$ be a continuous nondecreasing function such that $f(0) = 0$ and $\int_0^1 (F(t))^{-1/2} dt = +\infty$, where $F(t) = \int_0^t f(s)ds$. Under these assumptions, Vázquez proved that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{aligned} -\Delta u + f(u) &\geq 0 \quad \text{in } \Omega, \\ u &\geq 0 \quad \text{in } \Omega, \end{aligned} \tag{1.4}$$

then either $u \equiv 0$ in Ω or $u > 0$ in Ω. The growth assumption $\int_0^1 (F(t))^{-1/2} dt = +\infty$ shows that the Vázquez maximum principle holds true for "superlinear" nonlinearities; for instance, nonlinearities $f(u) = u^p$ with $p \geq 1$ satisfy the above hypotheses.

For details on the maximum principle we refer to the works by Protter and Weinberger [178], Gilbarg and Trudinger [99], Pucci and Serrin [181–183], Du [69], Fraenkel [81], and so forth.

1.2. Method of lower and upper solutions

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and consider a continuous function $f(x, u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that f is of class C^1 with respect to the variable u.

Consider the nonlinear Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= f(x, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

By the solution of problem (1.5), we mean a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ which satisfies (1.5).

We point out that the arguments we develop in what follows can be extended for solving more general nonlinear elliptic problems of the type

$$\begin{aligned} -\Delta u &= f(x, u) \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $g : \partial\Omega \to \mathbb{R}$ is continuous. By [99, Theorem 2.14], problem (1.6) corresponding to $f = 0$ has a solution, for every $g \in C(\partial\Omega)$, if and only if all boundary points of Ω are regular. That is why we assume throughout this chapter that Ω is a bounded domain in $\mathbb{R}^N$ and every point of $\partial\Omega$ is regular.

Definition 1.1. A function $\underline{U} \in C^2(\Omega) \cap C(\overline{\Omega})$ is said to be subsolution (lower solution) of problem (1.5) provided that

$$\begin{aligned} -\Delta \underline{U} &\le f(x,\underline{U}) \quad \text{in } \Omega,\\ \underline{U} &\le 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

Accordingly, if the signs are reversed in the above inequalities, we obtain the definition of a supersolution (upper solution) $\overline{U}$ for problem (1.5).

Theorem 1.2. *Let $\underline{U}$ (resp., $\overline{U}$) be a subsolution (resp., a supersolution) to problem* (1.5) *such that $\underline{U} \le \overline{U}$ in Ω. Then the following properties hold true:*

(i) *there exists a solution u of* (1.5) *satisfying $\underline{U} \le u \le \overline{U}$;*

(ii) *there exist minimal and maximal solutions $\underline{u}$ and $\overline{u}$ of problem* (1.5) *with respect to the interval $[\underline{U}, \overline{U}]$.*

Remark 1.3. The existence of the solution in this theorem, as well as the maximality (resp., minimality) of solutions given by (ii), should be understood with respect to the given pairing of ordered sub- and supersolutions. It is very possible that (1.5) has solutions which are *not* in the interval $[\underline{U}, \overline{U}]$. It may also happen that (1.5) has *not* maximal or minimal solution. Give such an example!

Remark 1.4. The hypothesis $\underline{U} \le \overline{U}$ is *not* automatically fulfilled for arbitrary sub- and supersolution of (1.5). Moreover, it may occur that $\underline{U} > \overline{U}$ on the *whole* domain Ω. An elementary example is the following: consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda_1 u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.8}$$

We know that all solutions of this problem are of the form $u = Ce_1$, where C is a real constant and e_1 is not vanishing in Ω, say $e_1(x) > 0$, for any $x \in \Omega$. Choose $\underline{U} = e_1$ and $\overline{U} = -e_1$. Then $\underline{U}$ (resp., $\overline{U}$) is subsolution (resp., supersolution) to the problem (1.5), but $\underline{U} > \overline{U}$.

Proof of Theorem 1.2. The key arguments are based on the monotone iterations method developed by Amann [9] and Sattinger [205].

(i) Let $g(x,u) := f(x,u) + au$, where a is a real constant. We can choose $a \ge 0$ sufficiently large so that the map $\mathbb{R} \ni u \mapsto g(x,u)$ is increasing on $[\underline{U}(x), \overline{U}(x)]$, for every $x \in \Omega$. For this aim, it is enough to have $a \ge 0$ and

$$a \ge \max\{-f_u(x,u); x \in \overline{\Omega},\ u \in [\underline{U}(x), \overline{U}(x)]\}. \tag{1.9}$$

For this choice of a, we define the sequence of functions $u_n \in C^2(\Omega) \cap C(\overline{\Omega})$ as follows: $u_0 = \overline{U}$ and, for every $n \ge 1$, u_n is the unique solution of the linear problem

$$\begin{aligned} -\Delta u_n + a u_n &= g(x, u_{n-1}) \quad \text{in } \Omega,\\ u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

Claim 1. $\underline{U} \le \cdots \le u_{n+1} \le u_n \le \cdots \le u_0 = \overline{U}$.

Proof of Claim 1. Our arguments use in an essential manner the maximum principle. So, in order to prove that $u_1 \le \overline{U}$, we have, by the definition of u_1,

$$\begin{gathered} -\Delta(\overline{U} - u_1) + a(\overline{U} - u_1) \ge g(x, \overline{U}) - g(x, \overline{U}) = 0 \quad \text{in } \Omega, \\ \overline{U} - u_1 \ge 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.11}$$

Since the operator $-\Delta + aI$ is coercive, it follows that $\overline{U} \ge u_1$ in Ω. For the proof of $\underline{U} \le u_1$, we observe that $\underline{U} \le 0 = u_1$ on $\partial\Omega$ and, for every $x \in \Omega$,

$$-\Delta(\underline{U} - u_1) + a(\underline{U} - u_1) \le f(x, \underline{U}) + a\underline{U} - g(x, \overline{U}) \le 0, \tag{1.12}$$

by the monotonicity of g. The maximum principle implies $\underline{U} \le u_1$.

Let us now assume that

$$\underline{U} \le \cdots \le u_n \le u_{n-1} \le \cdots \le u_0 = \overline{U}. \tag{1.13}$$

It remains to prove that

$$\underline{U} \le u_{n+1} \le u_n. \tag{1.14}$$

Taking into account the equations satisfied by u_n and u_{n+1}, we obtain

$$\begin{gathered} -\Delta(u_n - u_{n+1}) + a(u_n - u_{n+1}) = g(x, u_{n-1}) - g(x, u_n) \ge 0 \quad \text{in } \Omega, \\ u_n - u_{n+1} \ge 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.15}$$

which implies $u_n \ge u_{n+1}$ in Ω.

On the other hand, by

$$\begin{gathered} -\Delta\underline{U} + a\underline{U} \le g(x, \underline{U}) \quad \text{in } \Omega, \\ \underline{U} \le 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.16}$$

and the definition of u_{n+1}, we have

$$\begin{gathered} -\Delta(u_{n+1} - \underline{U}) + a(u_{n+1} - \underline{U}) \ge g(x, u_n) - g(x, \underline{U}) \ge 0 \quad \text{in } \Omega, \\ u_{n+1} - \underline{U} \ge 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.17}$$

Again, by the maximum principle, we deduce that $\underline{U} \le u_{n+1}$ in Ω, which completes the proof of the claim. □

It follows that there exists a function u such that, for every fixed $x \in \Omega$,

$$u_n(x) \searrow u(x) \quad \text{as } n \longrightarrow \infty. \tag{1.18}$$

Our aim is to show that we can pass to the limit in (1.10). For this aim, we use a standard bootstrap argument. Let $g_n(x) := g(x, u_n(x))$. We first observe that the sequence (g_n) is bounded in $L^\infty(\Omega)$, so in every $L^p(\Omega)$ with $1 < p < \infty$. It follows by (1.10) and standard

Schauder estimates that the sequence (u_n) is bounded in $W^{2,p}(\Omega)$, for any $1 < p < \infty$. But the space $W^{2,p}(\Omega)$ is continuously embedded in the Hölder space $C^{1,\alpha}(\overline{\Omega})$, for $\alpha = 1 - N/(2p)$, provided that $p > N/2$. This implies that (u_n) is bounded in $C^{1,\alpha}(\overline{\Omega})$. Now, by standard estimates in Hölder spaces, we deduce that (u_n) is bounded in $C^{2,\alpha}(\overline{\Omega})$. Since $C^{2,\alpha}(\overline{\Omega})$ is compactly embedded in $C^2(\overline{\Omega})$, it follows that, passing eventually at a subsequence,

$$u_n \longrightarrow u \qquad \text{in } C^2(\overline{\Omega}). \tag{1.19}$$

Since the sequence is monotone, we obtain that the whole sequence converges to u in C^2. Now, passing at the limit in (1.10) as $n \to \infty$ we deduce that u is solution of problem (1.5).

(ii) Let us denote by $\overline{u}$ the solution obtained with the above technique and choosing $u_0 = \overline{U}$. We justify that $\overline{u}$ is a maximal solution with respect to the given pairing $(\underline{U}, \overline{U})$. Indeed, let $u \in [\underline{U}, \overline{U}]$ be an arbitrary solution. With an argument similar to that given in the proof of (i) but with respect to the pairing of ordered sub- and supersolution $(u, \overline{U})$, we obtain that $u \leq u_n$, for any $n \geq 0$, which implies $u \leq \overline{u}$. □

The notions of lower and upper solutions can be extended in a *weak* sense as follows.

Definition 1.5. A function $\underline{U} \in C(\overline{\Omega})$ is said to be a weak subsolution (weak lower solution) of problem (1.5) provided that $\underline{U} \leq 0$ on $\partial\Omega$ and

$$\int_\Omega [\underline{U}(-\Delta\varphi) - f(x, \underline{U})\varphi]\,dx \leq 0, \tag{1.20}$$

for every $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi \geq 0$ in Ω.

A function $\overline{U} \in C(\overline{\Omega})$ is said to be a weak supersolution (weak upper solution) of problem (1.5) provided that $\overline{U} \geq 0$ on $\partial\Omega$ and

$$\int_\Omega [\overline{U}(-\Delta\varphi) - f(x, \overline{U})\varphi]\,dx \geq 0, \tag{1.21}$$

for every $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi \geq 0$ in Ω.

In such a case, the corresponding notion of solution for problem (1.5) is the following.

Definition 1.6. A function $u \in C(\overline{\Omega})$ is said to be a weak solution of problem (1.5) provided that $u = 0$ on $\partial\Omega$ and

$$\int_\Omega [u(-\Delta\varphi) - f(x, u)\varphi]\,dx = 0, \tag{1.22}$$

for every $\varphi \in C_0^\infty(\Omega)$.

Theorem 1.7. *Assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that the mapping $u \mapsto f(\cdot, u) + au$ is increasing, for some real number a. Suppose that $\underline{U}$ (resp., $\overline{U}$) is a weak* *subsolution (resp., a weak supersolution) of problem* (1.5) *such that* $\underline{U} \le \overline{U}$ *in* Ω. *Then the following properties hold true:*

(i) *there exists a weak solution* u *of* (1.5) *satisfying* $\underline{U} \le u \le \overline{U}$ *and* $u \in W^{2,p}_{\text{loc}}(\Omega)$ *for all* $p \in [1,\infty)$;

(ii) *there exist minimal and maximal weak solutions* $\underline{u}$ *and* $\overline{u}$ *of problem* (1.5) *with respect to the interval* $[\underline{U}, \overline{U}]$.

The proof follows the same steps as in the case where f is of class C^1. We invite the reader to give complete details.

1.3. An alternative proof without monotone iterations

Let us now assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, so *no differentiability assumption* is required to be fulfilled by f. We also point out that we do *not* assume that the mapping $u \mapsto f(\cdot, u) + au$ is increasing, for some real number a. Under these general hypotheses, we give in what follows an elementary proof of Theorem 1.7, by means of different arguments. For this aim, let

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx \tag{1.23}$$

be the energy functional associated to problem (1.5), where $F(x,u) = \int_0^u f(x,t)dt$.

Set

$$f_0(x,t) = \begin{cases} f(x,t) & \text{if } \underline{U}(x) < t < \overline{U}(x),\ x \in \overline{\Omega}, \\ f(x, \overline{U}(x)) & \text{if } t \ge \overline{U}(x),\ x \in \overline{\Omega}, \\ f(x, \underline{U}(x)) & \text{if } t \le \underline{U}(x),\ x \in \overline{\Omega}. \end{cases} \tag{1.24}$$

Then $f_0 : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

The associated energy functional is

$$E_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_0(x,u)dx, \tag{1.25}$$

with an appropriate definition for F_0.

We observe the following:

(i) E_0 is well defined on $H^1(\Omega)$, since f_0 is bounded, so F_0 has a sublinear growth;

(ii) E_0 is weakly lower semicontinuous;

(iii) the first term of E_0 dominates at $+\infty$, in the sense that

$$\lim_{\|u\| \to \infty} E_0(u) = +\infty. \tag{1.26}$$

Let

$$\alpha = \inf_{u \in H^1_0(\Omega)} E_0(u). \tag{1.27}$$

We show in what follows that α is attained. Indeed, since E_0 is coercive, there exists a minimizing sequence $(u_n) \subset H_0^1(\Omega)$. We may assume without loss of generality that

$$u_n \rightharpoonup u, \quad \text{weakly in } H_0^1(\Omega). \tag{1.28}$$

So, by the lower semicontinuity of E_0 with respect to the weak topology,

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_\Omega F_0(x, u_n)\,dx \le \alpha + o(1). \tag{1.29}$$

This implies $E_0(u) = \alpha$. Now, since u is a critical point of E_0, it follows that it satisfies $-\Delta u = f_0(x, u)$ in $\mathcal{D}'(\Omega)$, that is,

$$\int_\Omega [u(-\Delta\varphi) - f_0(x, u)\varphi]\,dx = 0, \tag{1.30}$$

for every $\varphi \in C_0^\infty(\Omega)$.

The same bootstrap argument as in the proof of Theorem 1.2 implies that $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all $p \in [1, \infty)$, although $\underline{U}$ and $\overline{U}$ do not need to possess the same regularity.

We prove in what follows that $\underline{U} \le u \le \overline{U}$. Indeed, we have

$$-\Delta \underline{U} \le f(x, \underline{U}) \quad \text{in } \mathcal{D}'(\Omega). \tag{1.31}$$

Therefore,

$$-\Delta(\underline{U} - u) \le f(x, \underline{U}) - f_0(x, u) \quad \text{in } \mathcal{D}'(\Omega). \tag{1.32}$$

After multiplication by $(\underline{U} - u)^+$ in this inequality and integration over Ω, we find

$$\int_\Omega \left|\nabla(\underline{U} - u)^+\right|^2 dx \le \int_\Omega (\underline{U} - u)^+ (f(x, \underline{U}) - f_0(x, u))\,dx. \tag{1.33}$$

Taking into account the definition of f_0, we obtain

$$\int_\Omega \left|\nabla(\underline{U} - u)^+\right|^2 dx = 0 \tag{1.34}$$

which implies $\nabla(\underline{U} - u)^+ = 0$ in Ω. Therefore, $\underline{U} \le u$ in Ω and, similarly, $u \le \overline{U}$ in Ω. So, since u is a weak solution of the problem

$$\begin{gathered} -\Delta u = f_0(x, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.35}$$

and $\underline{U} \le u \le \overline{U}$ in Ω, then u is a solution of problem (1.5).

To show that the problem $-\Delta u = f_0(x, u)$ in $\mathcal{D}'(\Omega)$ has at least one solution, we give in what follows an alternative argument due to Clement and Sweers [48], which relies upon the Schauder fixed point theorem. We first observe that for any continuous function $h : \overline{\Omega} \to \mathbb{R}$, the *linear* boundary value problem

$$\begin{gathered} -\Delta u = h(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.36}$$

has a unique solution $u \in C(\overline{\Omega})$. Let $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ denote the solution operator defined by $u = Kh$. Then K is a linear operator.

Claim 2. K is a compact operator, provided the metric space $C(\overline{\Omega})$ is endowed with the canonical maximum norm

$$\|u\|_\infty = \max_{x\in\overline{\Omega}} |u(x)|. \tag{1.37}$$

For this purpose we first extend h by 0 outside of $\overline{\Omega}$ and set

$$w(x) = -\int_{\mathbb{R}^N} \Gamma(x-y)h(y)dy, \tag{1.38}$$

where

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi}\log|x| & \text{if } N = 2,\ x \neq 0, \\ \dfrac{1}{(2-N)\omega_N}|x|^{2-N} & \text{if } N \geq 3,\ x \neq 0 \end{cases} \tag{1.39}$$

denotes the fundamental solution of the Laplace equation. In the definition of Γ, ω_N represents the outer Lebesgue measure of the unit ball in $\mathbb{R}^N$. Standards results in the linear theory of elliptic equations show that $w \in C^1(\overline{\Omega})$ and the mapping $h \mapsto w$ from $C(\overline{\Omega})$ to $C^1(\overline{\Omega})$ is continuous, where $C^1(\overline{\Omega})$ is endowed with the norm $\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty$. Since $C^1(\overline{\Omega})$ is compactly embedded in $C(\overline{\Omega})$, we deduce that the mapping

$$C(\overline{\Omega}) \ni h \longmapsto w \in C(\overline{\Omega}) \tag{1.40}$$

is compact.

Let $w_0 \in C(\overline{\Omega})$ be the unique harmonic function satisfying $w_0 = w$ on $\partial\Omega$. Then the function $u = w - w_0$ is continuous on $\overline{\Omega}$, $u = 0$ on $\partial\Omega$, and

$$\int_\Omega [u(-\Delta\varphi) - h(x)\varphi]dx = 0, \tag{1.41}$$

for every $\varphi \in C_0^\infty(\Omega)$. Since the mapping $w \mapsto w_0$ from $C(\overline{\Omega})$ to $C(\overline{\Omega})$ is continuous, we deduce that the mapping $h \mapsto w_0$ from $C(\overline{\Omega})$ to $C(\overline{\Omega})$ is compact. This concludes the proof of the claim.

Returning to our proof, we denote by $\mathcal{N} : C(\overline{\Omega}) \to C(\overline{\Omega})$ the Niemytski operator associated with f_0, that is,

$$\mathcal{N}(u)(x) = f_0(x,u(x)) \quad \text{for } u \in C(\overline{\Omega}),\ x \in \overline{\Omega}. \tag{1.42}$$

Then $\mathcal{N}$ is continuous and, since f_0 is bounded, there exists $C > 0$ such that

$$||\mathcal{N}(u)||_\infty \leq C \quad \forall u \in C(\overline{\Omega}). \tag{1.43}$$

Finally, we observe that u is a solution of the problem

$$\begin{aligned} -\Delta u &= f_0(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.44}$$

if and only if

$$u = K\mathcal{N}(u). \tag{1.45}$$

A direct application of the Schauder fixed point theorem guarantees the existence of such a solution.

1.4. Stability of solutions

We can interpret a solution u of problem (1.5) as an equilibrium solution of the associated parabolic problem

$$\begin{gathered} v_t - \Delta v = f(x,v) \quad \text{in } \Omega \times (0,\infty), \\ v(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) = u_0(x) \quad \text{in } \Omega. \end{gathered} \tag{1.46}$$

Suppose that the initial data $u_0(x)$ does not deviate too much from a stationary state $u(x)$. Does the solution of (1.46) return to $u(x)$ as $t \to \infty$? If this is the case, then the solution u of problem (1.5) is said to be stable. More precisely, a solution u of (1.46) is called *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u(x) - v(x,t)\|_{L^\infty(\Omega\times(0,\infty))} < \varepsilon$, provided that $\|u(x) - u_0(x)\|_{L^\infty(\Omega)} < \delta$. Here, $v(x,t)$ is a solution of problem (1.46). We establish in what follows that the solutions given by the method of lower and upper solutions are, generally, stable, in the following sense.

Definition 1.8. A solution u of problem (1.5) is said to be stable provided that the first eigenvalue of the linearized operator at u is positive, that is, $\lambda_1(-\Delta - f_u(x,u)) > 0$.

A solution u of (1.5) is called semistable if $\lambda_1(-\Delta - f_u(x,u)) \geq 0$.

In the above definition, we understand the first eigenvalue of the linearized operator with respect to homogeneous Dirichlet boundary condition. This means that

$$\lambda_1(-\Delta - f_u(x,u)) = \min_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla v|^2 - f_u(x,u)v^2)dx}{\int_\Omega v^2 dx}. \tag{1.47}$$

Theorem 1.9. *Let $\underline{U}$ (resp., $\overline{U}$) be subsolution (resp., supersolution) of problem* (1.5) *such that $\underline{U} \leq \overline{U}$ and let $\underline{u}$ (resp., $\overline{u}$) be the corresponding minimal (resp., maximal) solution of* (1.5)*. Assume that $\underline{U}$ is not a solution of* (1.5)*. Then $\underline{u}$ is semistable. Furthermore, if f is concave, then $\underline{u}$ is stable.*

Similarly, if $\overline{U}$ is not a solution then $\overline{u}$ is semistable and, if f is convex, then $\overline{u}$ is stable.

Proof. Let $\lambda_1 = \lambda_1(-\Delta - f_u(x,\underline{u}))$ and let φ_1 be the corresponding eigenfunction, that is,

$$\begin{gathered} -\Delta\varphi_1 - f_u(x,\underline{u})\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega, \\ \varphi_1 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.48}$$

We can suppose, without loss of generality, that $\varphi_1 > 0$ in Ω. Assume, by contradiction, that $\lambda_1 < 0$. Let us consider the function $v := \underline{u} - \varepsilon\varphi_1$, with $\varepsilon > 0$. Next, we prove that the following properties hold true:

(i) v is a supersolution to problem (1.5), for ε small enough;

(ii) $v \geq \underline{U}$.

By (i), (ii), and Theorem 1.2 it follows that there exists a solution u such that $\underline{U} \leq u \leq v < \underline{u}$ in Ω, which contradicts the minimality of $\underline{u}$ and the hypothesis that $\underline{U}$ is not a solution.

In order to prove (i), it is enough to show that

$$-\Delta v \geq f(x, v) \quad \text{in } \Omega. \tag{1.49}$$

But

$$\begin{aligned}
\Delta v + f(x, v) &= \Delta\underline{u} - \varepsilon\Delta\varphi_1 + f(x, \underline{u} - \varepsilon\varphi_1) \\
&= -f(x, \underline{u}) + \varepsilon\lambda_1\varphi_1 + \varepsilon f_u(x, \underline{u})\varphi_1 + f(x, \underline{u} - \varepsilon\varphi_1) \\
&= -f(x, \underline{u}) + \varepsilon(\lambda_1\varphi_1 + f_u(x, \underline{u})\varphi_1) + f(x, \underline{u}) - \varepsilon f_u(x, \underline{u})\varphi_1 + o(\varepsilon\varphi_1) \\
&= \varepsilon\lambda_1\varphi_1 + o(\varepsilon)\varphi_1 = \varphi_1(\varepsilon\lambda_1 + o(\varepsilon)) = \varphi_1\varepsilon(\lambda_1 + o(1)) \leq 0,
\end{aligned} \tag{1.50}$$

provided that $\varepsilon > 0$ is sufficiently small.

Let us now prove (ii). We observe that $v \geq \underline{U}$ is equivalent to $\underline{u} - \underline{U} \geq \varepsilon\varphi_1$, for small positive ε. But $\underline{u} - \underline{U} \geq 0$ in Ω. Moreover, $\underline{u} - \underline{U} \not\equiv 0$, since $\underline{U}$ is not solution. Now we are in position to apply Hopf's strong maximum principle in the following variant: assume v satisfies

$$\begin{gathered}
-\Delta w + aw = f(x) \geq 0 \quad \text{in } \Omega, \\
w \geq 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.51}$$

where a is a nonnegative number. Then $w \geq 0$ in Ω and the following alternative holds:

(i) either $w \equiv 0$ in Ω, or

(ii) $w > 0$ in Ω and $\partial w/\partial \nu < 0$ on the set $\{x \in \partial\Omega;\ w(x) = 0\}$.

Let $w = \underline{u} - \underline{U} \geq 0$. We have

$$-\Delta w + aw = f(x, \underline{u}) + \Delta\underline{U} + a(\underline{u} - \underline{U}) \geq f(x, \underline{u}) - f(x, \underline{U}) + a(\underline{u} - \underline{U}). \tag{1.52}$$

So, in order to have $-\Delta w + aw \geq 0$ in Ω, it is sufficient to choose $a \geq 0$ so that the mapping $\mathbb{R} \ni u \mapsto f(x, u) + au$ is increasing on $[\underline{U}(x), \overline{U}(x)]$, as already done in the proof of Theorem 1.2. Observing that $w \geq 0$ on $\partial\Omega$ and $w \not\equiv 0$ in Ω we deduce by the strong maximum principle that

$$\begin{gathered}
w > 0 \quad \text{in } \Omega, \\
\frac{\partial w}{\partial \nu} < 0 \quad \text{on } \{x \in \partial\Omega;\ \underline{u}(x) = \underline{U}(x) = 0\}.
\end{gathered} \tag{1.53}$$

We prove in what follows that we can choose $\varepsilon > 0$ sufficiently small so that $\varepsilon\varphi_1 \leq w$. This is an interesting consequence of the fact that the normal derivative is negative in the points of the boundary where the function vanishes. Arguing by contradiction, there exist a sequence $\varepsilon_n \to 0$ and $x_n \in \Omega$ such that

$$(w - \varepsilon_n \varphi_1)(x_n) < 0. \tag{1.54}$$

Moreover, we can choose the points x_n with the additional property

$$\nabla (w - \varepsilon_n \varphi_1)(x_n) = 0. \tag{1.55}$$

But, passing eventually at a subsequence, we can assume that $x_n \to x_0 \in \overline{\Omega}$. It follows now by (1.54) that $w(x_0) \leq 0$ which implies $w(x_0) = 0$, that is, $x_0 \in \partial\Omega$. Furthermore, by (1.55), $\nabla w(x_0) = 0$, a contradiction, since $\partial w/\partial \nu(x_0) < 0$, by the strong maximum principle.

Let us now assume that f is concave. We have to show that $\lambda_1 > 0$. Arguing again by contradiction, let us suppose that $\lambda_1 = 0$. With the same arguments as above we can show that $v \geq \underline{U}$. If we prove that v is a supersolution then we contradict the minimality of $\underline{u}$. The above arguments do not apply since, in order to find a contradiction, the estimate

$$\Delta v + f(x, v) = \varepsilon \varphi_1 (\lambda_1 + o(1)) \tag{1.56}$$

is not relevant in the case where $\lambda_1 = 0$. However,

$$\begin{aligned} \Delta v + f(x, v) &= -f(x, \underline{u}) + \varepsilon(\lambda_1 \varphi_1 + f_u(x, \underline{u})\varphi_1) + f(x, \underline{u} - \varepsilon\varphi_1) \\ &\leq \varepsilon f_u(x, \underline{u})\varphi_1 + f_u(x, \underline{u})(- \varepsilon\varphi_1) \\ &= 0. \end{aligned} \tag{1.57}$$

This completes our proof. □

If neither $\underline{U}$ nor $\overline{U}$ are solutions to problem (1.5), it is natural to ask us if there exists a solution u such that $\underline{U} < u < \overline{U}$ and $\lambda_1(-\Delta - f_u(x, u)) > 0$. In general, such a situation does not occur, as shown by the following example: consider the problem

$$\begin{aligned} -\Delta u &= \lambda_1 u - u^3 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.58}$$

where λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. We remark that we can choose $\overline{U} = a$ and $\underline{U} = -a$, for every $a > \sqrt{\lambda_1}$. On the other hand, by Poincaré's inequality,

$$\lambda_1 \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 = \lambda_1 \int_\Omega u^2 - \int_\Omega u^4, \tag{1.59}$$

which shows that the unique solution is $u = 0$. However, this solution is not stable, since $\lambda_1(-\Delta - f_u(0)) = 0$.

1.5. Extremal solutions

A question which arises naturally is under what hypotheses there exists a *global* maximal (resp., minimal) solution of (1.5), that is, which is maximal (resp., minimal) not only with respect to a prescribed pair of sub- and supersolutions. The following result shows that a sufficient condition is that the nonlinearity has a kind of sublinear growth. More precisely, let us consider the problem

$$\begin{aligned} -\Delta u &= f(x,u) + g(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.60}$$

Theorem 1.10. *Assume that* $g \in C^{0,\alpha}(\overline{\Omega})$, *for some* $\alpha \in (0,1)$, *and there exists* $a < \lambda_1$ *such that for every* $(x,u) \in \Omega \times \mathbb{R}$,

$$f(x,u)\operatorname{sign} u \leq a|u| + C. \tag{1.61}$$

Then there exists a global minimal (resp., maximal) solution $\underline{u}$ *(resp.,* $\overline{u}$*) for problem* (1.60).

Proof. Assume without loss of generality that $C > 0$. We choose as supersolution of (1.60) the unique solution $\overline{U}$ of the linear coercive problem

$$\begin{aligned} -\Delta\overline{U} - a\overline{U} &= C' \quad \text{in } \Omega, \\ \overline{U} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.62}$$

where C' is taken such that $C' \geq C + \sup_{\overline{\Omega}} |g|$. Since $a < \lambda_1$, it follows by the maximum principle that $\overline{U} \geq 0$.

Let $\underline{U} = -\overline{U}$ be a subsolution of (1.60). Thus, by Theorem 1.2, there exists $\underline{u}$ (resp., $\overline{u}$) minimal (resp., maximal) with respect to $(\underline{U}, \overline{U})$. We prove in what follows that $\underline{u} \leq u \leq \overline{u}$, for *every* solution u of problem (1.60). For this aim, it is enough to show that $\underline{U} \leq u \leq \overline{U}$. Let us prove that $u \leq \overline{U}$. Denote

$$\Omega_0 = \{x \in \Omega;\ u(x) > 0\}. \tag{1.63}$$

Consequently, it is sufficient to show that $u \leq \overline{U}$ in Ω_0. The idea is to prove that

$$\begin{aligned} -\Delta(\overline{U} - u) - a(\overline{U} - u) &\geq 0 \quad \text{in } \Omega_0, \\ \overline{U} - u &\geq 0 \quad \text{on } \partial\Omega_0 \end{aligned} \tag{1.64}$$

and then to apply the maximum principle. On the one hand, we observe that

$$\overline{U} - u = \overline{U} \geq 0 \quad \text{on } \partial\Omega_0. \tag{1.65}$$

On the other hand, we have

$$-\Delta(\overline{U} - u) - a(\overline{U} - u) = -\Delta\overline{U} - a\overline{U} - (-\Delta u - au) \geq C' - f(x,u) + au \geq 0 \quad \text{in } \Omega_0. \tag{1.66}$$

This concludes our proof. □

Let us now consider the nonlinear boundary value problem

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \tag{1.67}$$

where

$$f(0) = 0, \tag{1.68}$$

$$\limsup_{u \to +\infty} \frac{f(u)}{u} < \lambda_1. \tag{1.69}$$

Observe that (1.69) implies

$$f(u) \leq au + C, \quad \text{for every } u \geq 0, \tag{1.70}$$

with $a < \lambda_1$ and $C > 0$.

Clearly, $\underline{U} = 0$ is a subsolution of (1.67). Let $\overline{U}$ be the unique solution of the linear coercive problem

$$\begin{gathered} -\Delta\overline{U} - a\overline{U} = C \quad \text{in } \Omega, \\ \overline{U} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.71}$$

We then obtain a minimal solution $\underline{u}$ and a maximal solution $\overline{u} \geq 0$. However, we cannot state that $\overline{u} > 0$ (give an example!). A positive answer is given in the following result.

Theorem 1.11. *Assume f satisfies hypotheses* (1.68), (1.69), *and*

$$f'(0) > \lambda_1. \tag{1.72}$$

Then there exists a maximal solution u to problem (1.67) *such that $u > 0$ in Ω.*

Proof. The idea is to find another subsolution. Let $\underline{U} = \varepsilon\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. To obtain our conclusion it is sufficient to verify that, for $\varepsilon > 0$ small enough, we have

(i) $\varepsilon\varphi_1$ is a subsolution;
(ii) $\varepsilon\varphi_1 \leq \overline{U}$.

Let us verify (i). We observe that

$$f(\varepsilon\varphi_1) = f(0) + \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1) = \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1). \tag{1.73}$$

So the inequality $-\Delta(\varepsilon\varphi_1) \leq f(\varepsilon\varphi_1)$ is equivalent to

$$\varepsilon\lambda_1\varphi_1 \leq \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1), \tag{1.74}$$

that is,

$$\lambda_1 \leq f'(0) + o(1). \tag{1.75}$$

This is true, by our hypothesis (1.72).

Let us now verify (ii). Recall that $\overline{U}$ satisfies

$$
\begin{aligned}
-\Delta\overline{U} &= a\overline{U} + C \quad \text{in } \Omega, \\
\overline{U} &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{1.76}
$$

Thus, by the maximum principle, $\overline{U} > 0$ in Ω and $\partial\overline{U}/\partial\nu < 0$ on $\partial\Omega$. We observe that the other variant, namely, $\overline{U} \equiv 0$, becomes impossible, since $C > 0$. Using the same trick as in the proof of Theorem 1.2 (more precisely, the fact that $\partial\overline{U}/\partial\nu < 0$ on $\partial\Omega$), we find $\varepsilon > 0$ small enough so that $\varepsilon\varphi_1 \leq \overline{U}$ in Ω. □

Remark 1.12. We observe that a necessary condition for the existence of a solution to problem (1.67) is that the line $\lambda_1 u$ intersects the graph of the function $f = f(u)$ on the positive semiaxis.

Indeed, if $f(u) < \lambda_1 u$ for any $u > 0$, then the unique solution is $u = 0$. After multiplication with φ_1 in (1.67) and integration, we find

$$
\int_\Omega (-\Delta u)\varphi_1 = -\int_\Omega u\Delta\varphi_1 = \lambda_1 \int_\Omega u\varphi_1 = \int_\Omega f(u)\varphi_1 < \lambda_1 \int_\Omega u\varphi_1, \tag{1.77}
$$

a contradiction.

Remark 1.13. We can require instead of (1.72) that $f \in C^1(0,\infty)$ and $f'(0+) = +\infty$.

Indeed, since $f(0) = 0$, there exists $c > 0$ such that, for all $0 < \varepsilon < c$,

$$
\frac{f(\varepsilon\varphi_1)}{\varepsilon\varphi_1} = \frac{f(\varepsilon\varphi_1) - f(0)}{\varepsilon\varphi_1 - 0} > \lambda_1. \tag{1.78}
$$

It follows that $f(\varepsilon\varphi_1) > \varepsilon\lambda_1\varphi_1 = -\Delta(\varepsilon\varphi_1)$ and so, $\underline{U} = \varepsilon\varphi_1$ is a subsolution. It is easy to check that $\overline{U} \geq \underline{U}$ in Ω and the proof continues with the same ideas as above.

The following result gives a sufficient condition such that the solution of (1.67) is unique.

Theorem 1.14. *Under hypotheses* (1.68), (1.69), *and* (1.72), *assume furthermore that the mapping*

$$
(0,+\infty) \ni u \longmapsto \frac{f(u)}{u} \tag{1.79}
$$

is decreasing. Then problem (1.67) *has a unique solution.*

Example 1.15. If f is concave, then the mapping $f(u)/u$ is decreasing. Hence by the previous results, the solution is unique and stable. For instance, the problem

$$
\begin{aligned}
-\Delta u &= \lambda u - u^p \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega, \\
u &> 0 \quad \text{in } \Omega,
\end{aligned}
\tag{1.80}
$$

with $p > 1$ and $\lambda > \lambda_1$, has a unique solution which is also stable.

Proof of Theorem 1.14. Let u_1, u_2 be arbitrary solutions of (1.67). We may assume that $u_1 \leq u_2$; indeed, if not, we choose u_1 as the minimal solution. Multiplying the equalities

$$-\Delta u_1 = f(u_1) \quad \text{in } \Omega,$$
$$-\Delta u_2 = f(u_2) \quad \text{in } \Omega \tag{1.81}$$

by u_2, u_1, respectively, and integrating on Ω, we find

$$\int_\Omega \left(f(u_1)u_2 - f(u_2)u_1\right) = 0. \tag{1.82}$$

This relation can be rewritten as

$$\int_\Omega u_1 u_2 \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right) = 0. \tag{1.83}$$

So, by $0 < u_1 \leq u_2$ we deduce $f(u_1)/u_1 = f(u_2)/u_2$ in Ω. Next, by (1.79), we conclude that $u_1 = u_2$. □

1.6. The Krasnoselski uniqueness theorem

Consider the nonlinear Dirichlet problem

$$-\Delta u = f(u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.84}$$
$$u > 0 \quad \text{in } \Omega.$$

Our aim is to establish in what follows a uniqueness property in the case where the nonlinearity does not satisfy any growth assumptions, like (1.69) or (1.72). The following important result in this direction is due to Krasnoselski.

Theorem 1.16. *Assume that the nonlinearity f satisfies* (1.79). *Then problem* (1.84) *has a unique solution.*

Proof. In order to prove the uniqueness, it is enough to show that for any arbitrary solutions u_1 and u_2, we can suppose that $u_1 \leq u_2$. Let

$$A = \{t \in [0,1];\ tu_1 \leq u_2\}. \tag{1.85}$$

We observe that $0 \in A$. Next, we show that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have $\varepsilon \in A$. This follows easily with arguments which are already done and using the crucial observations that

$$u_2 > 0 \quad \text{in } \Omega,$$
$$\frac{\partial u_2}{\partial \nu} < 0 \quad \text{on } \partial\Omega. \tag{1.86}$$

Let $t_0 = \max A$. Assume, by contradiction, that $t_0 < 1$. Hence $t_0 u_1 \le u_2$ in Ω. The idea is to show the existence of some $\varepsilon > 0$ such that $(t_0 + \varepsilon)u_1 \le u_2$, which contradicts the choice of t_0. For this aim we use the maximum principle. We have

$$-\Delta(u_2 - t_0 u_1) + a(u_2 - t_0 u_1) = f(u_2) + a u_2 - t_0(f(u_1) + a u_1). \tag{1.87}$$

Now we choose $a > 0$ so that the mapping $u \mapsto f(u) + au$ is increasing. Therefore,

$$\begin{aligned}
-\Delta(u_2 - t_0 u_1) + a(u_2 - t_0 u_1) &\ge f(t_0 u_1) + a t_0 u_1 - t_0 f(u_1) - a t_0 u_1 \\
&= f(t_0 u_1) - t_0 f(u_1) \\
&\ge 0.
\end{aligned} \tag{1.88}$$

This implies

(i) either $u_2 - t_0 u_1 \equiv 0$, or

(ii) $u_2 - t_0 u_1 > 0$ in Ω and $\partial(u_2 - t_0 u_1)/\partial\nu < 0$ on $\partial\Omega$.

The first case is impossible since it would imply $t_0 f(u_1) = f(t_0 u_1)$, a contradiction. This reasoning is based on the elementary fact that if f is continuous and $f(Cx) = Cf(x)$ for any x in a nonempty interval, then f is linear. □

1.7. The Brezis-Oswald theorem

We establish in this section a basic result which is due to Brezis and Oswald (see [35, Theorem 1]) and which plays an important role in the qualitative analysis of wide classes of nonlinear Dirichlet boundary value problems.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that $g(x,u) : \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is a measurable function satisfying the following conditions:

$$\begin{gathered}
\text{for a.e. } x \in \Omega \text{ the function } [0,\infty) \ni u \longmapsto g(x,u) \\
\text{is continuous and the mapping } u \longmapsto \frac{g(x,u)}{u} \text{ is decreasing on } (0,\infty);
\end{gathered} \tag{1.89}$$

$$\text{for each } u \ge 0 \text{ the function } x \longmapsto g(x,u) \text{ belongs to } L^\infty(\Omega); \tag{1.90}$$

$$\exists C > 0 \text{ such that } g(x,u) \le C(u+1) \text{ a.e. } x \in \Omega \quad \forall u \ge 0. \tag{1.91}$$

Set

$$\begin{aligned}
a_0(x) &= \lim_{u \searrow 0} \frac{g(x,u)}{u}, \\
a_\infty(x) &= \lim_{u \to \infty} \frac{g(x,u)}{u},
\end{aligned} \tag{1.92}$$

so that $-\infty < a_0(x) \le +\infty$ and $-\infty \le a_\infty(x) < +\infty$.

For any measurable function $a : \Omega \to \mathbb{R}$ such that either $a(x) \le C$ or $a(x) \ge -C$ a.e. on Ω (for some positive constant C), the quantity

$$\int_{[\varphi \ne 0]} a\varphi^2 \tag{1.93}$$

makes sense, where $\varphi \in L^2(\Omega)$. In such a case, we can define the first eigenvalue of the operator $-\Delta - a(x)$ with zero Dirichlet condition by

$$\lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1(\Omega),\, \|\varphi\|_{L^2(\Omega)}=1} \left(\int |\nabla \varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right). \tag{1.94}$$

In our setting, $a_\infty(x) \le g(x,1)$ and $a_0(x) \ge g(x,1)$ for a.e. $x \in \Omega$. Thus, it makes sense to define both $\lambda_1(-\Delta - a_0(x))$ and $\lambda_1(-\Delta - a_\infty(x))$.

Consider the Dirichlet boundary value problem

$$\begin{gathered} -\Delta u = g(x,u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.95}$$

By a solution of problem (1.95), we mean a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying (1.95). Moreover, from assumptions (1.89)–(1.91) it follows that if u solves (1.95) then

$$-\left| g(x, \|u\|_{L^\infty}) \right| \le g(x, u(x)) \le C(|u(x)| + 1). \tag{1.96}$$

Thus, $g(x,u) \in L^\infty(\Omega)$ and standard Schauder estimates imply that $u \in W^{2,p}(\Omega)$ for every $p < \infty$.

Brezis and Oswald proved in [35] the following existence and uniqueness result. Our proof of existence relies on a variational method which is based on some ideas developed by Díaz and Saa [64].

Theorem 1.17. *Assume that hypotheses* (1.89)–(1.91) *are fulfilled.*

Then problem (1.95) *has at most one solution. Moreover, a solution of* (1.95) *exists if and only if*

$$\lambda_1(-\Delta - a_0(x)) < 0, \tag{1.97}$$

$$\lambda_1(-\Delta - a_\infty(x)) > 0. \tag{1.98}$$

Proof. We first assume that there exists a solution u of problem (1.95). Hence

$$\lambda_1(-\Delta - a_0(x)) \le \frac{1}{\int_\Omega u^2 dx} \left(\int_\Omega |\nabla u|^2 dx - \int_\Omega a_0 u^2 dx \right). \tag{1.99}$$

On the other hand, by Green's formula,

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega \Delta u \cdot u\, dx = \int_\Omega \frac{g(x,u)}{u} u^2 dx < \int_\Omega a_0 u^2 dx. \tag{1.100}$$

Thus, $\lambda_1(-\Delta - a_0(x)) < 0$.

Next, we show that $\lambda_1(-\Delta - a_\infty(x)) > 0$. Set

$$a(x) = \frac{g(x, 1 + \|u\|_\infty)}{1 + \|u\|_\infty}. \tag{1.101}$$

Thus, $a(x) \geq a_\infty(x)$ for a.e. $x \in \Omega$ and $g(x,u) \geq a(x)u$. We deduce that

$$\lambda_1\big(-\Delta - a_\infty(x)\big) > \lambda_1\big(-\Delta - a(x)\big) := \mu. \tag{1.102}$$

It remains to prove that $\mu > 0$. For this purpose, we consider the first eigenfunction $\Psi > 0$ of the linear operator $-\Delta - a(x)$ in $H_0^1(\Omega)$, that is,

$$\begin{gathered} -\Delta\Psi - a(x)\Psi = \mu\Psi \quad \text{in } \Omega, \\ \Psi = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.103}$$

Define, for any positive integer k,

$$\Omega_k := \{x \in \Omega;\ k\Psi(x) > u(x)\}. \tag{1.104}$$

Then $\Omega = \bigcup_{k\geq 1}\Omega_k$. So, by the definition of $a(x)$ and our assumption (1.89),

$$\int_{\Omega_k}\left(-\frac{\Delta u}{u} + \frac{\Delta(k\Psi)}{k\Psi}\right)(u - k\Psi)dx = \int_{\Omega_k}\left(\frac{g(x,u)}{u} - \big(a(x)+\mu\big)\right)(u-k\Psi)dx \geq 0. \tag{1.105}$$

It follows that $g(x,u)/u < a(x)+\mu$. Since $g(x,u)/u > a(x)$, we conclude that $\mu > 0$.

Conversely, let us assume that $\lambda_1(-\Delta - a_0(x)) < 0 < \lambda_1(-\Delta - a_\infty(x))$. We prove that problem (1.95) has a solution. Define the mapping

$$g_0(x,u) = \begin{cases} g(x,u) & \text{if } u \geq 0, \\ g(x,0) & \text{if } u < 0. \end{cases} \tag{1.106}$$

Set $G_0(x,u) = \int_0^u g_0(x,t)dt$ and consider the energy functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G_0(x,u)dx. \tag{1.107}$$

By our assumption (1.91) it follows that for all $(x,u) \in \Omega \times \mathbb{R}$,

$$\big|G_0(x,u)\big| \leq C\big(u^2 + |u|\big) \quad \text{for some } C > 0, \tag{1.108}$$

which implies that E is well defined on $H_0^1(\Omega)$. □

Lemma 1.18. *The energy functional E has the following properties:*

(a) *E is weakly lower semicontinuous in* $H_0^1(\Omega)$*;*
(b) *there exists* $\phi \in H_0^1(\Omega)$ *such that* $E(\phi) < 0$*;*
(c) *E is coercive, that is,* $\lim_{\|u\|\to\infty} E(u) = +\infty$.

Proof. (a) Let $(u_n)_{n\geq 1}$ be a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. We first observe that the lower semicontinuity of the norm yields

$$\|u\|^2_{H_0^1(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|^2_{H_0^1(\Omega)}. \tag{1.109}$$

We can suppose that, up to a subsequence, $(u_n)_{n\geq 1}$ converges strongly to u in $L^2(\Omega)$ and a.e. Since $|G_0(x,u)| \leq C(u^2+|u|)$ in $\Omega\times\mathbb{R}$, it follows that we can choose $C>0$ large enough so that $G_0(x,u_n) \leq C(1+u_n^2)$ for all $n\geq 1$ and for a.e. $x\in\Omega$. So, by Fatou's lemma,

$$\limsup_{n\to\infty}\int_\Omega G_0(x,u_n)\,dx \leq \int_\Omega G_0(x,u)\,dx. \tag{1.110}$$

Thus, $E(u)\leq \liminf_{n\to\infty} E(u_n)$.

(b) Since $\lambda_1(-\Delta - a_0(x))<0$, there exists $\phi\in H_0^1(\Omega)$ such that

$$|\nabla\phi|^2dx < \int_{[\phi\neq 0]} a_0\phi^2dx. \tag{1.111}$$

This relation remains valid if we replace ϕ by ϕ^+, so we may assume that $\phi\geq 0$.

Next, we show that we can assume $\phi\in L^\infty(\Omega)$. Set, for any positive integer k, $\Omega_k := \{x\in\Omega;\ \phi(x)\leq k\}$. Then $\Omega=\bigcup_{k\geq 1}\Omega_k$ and for any $k\geq 1$, $\Omega_k\subset\Omega_{k+1}$. Hence

$$\int_\Omega |\nabla\phi|^2dx = \lim_{n\to\infty}\int_{\Omega_k}|\nabla\phi|^2dx. \tag{1.112}$$

So, by relation (1.111), there is some positive integer k such that

$$\int_{\Omega_k}|\nabla\phi|^2dx < \int_{\Omega_k\cap[\phi\neq 0]} a_0\phi^2dx. \tag{1.113}$$

Fix k with such a property and denote

$$\varphi(x)=\begin{cases}\phi(x) & \text{if } \phi(x)\leq k,\\ 0 & \text{if } \phi(x)>k.\end{cases} \tag{1.114}$$

Hence $\varphi\in L^\infty(\Omega)$ and

$$\int_\Omega|\nabla\varphi|^2dx=\int_{\Omega_k}|\nabla\phi|^2dx<\int_{\Omega_k\cap[\phi\neq 0]}a_0\phi^2dx=\int_{[\varphi\neq 0]}a_0\phi^2dx. \tag{1.115}$$

This shows that we can replace ϕ by $\varphi\in L^\infty(\Omega)$. In this way, we have argued that we can choose from the beginning $\phi\in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $\phi\geq 0$.

From the definition of a_0 we deduce that

$$\liminf_{u\to 0}\frac{G_0(x,u)}{u^2}\geq\frac{a_0(x)}{2}. \tag{1.116}$$

Taking $u=\varepsilon\phi$ in this inequality, for $\varepsilon>0$ and ϕ as above, we have

$$\liminf_{\varepsilon\to 0}\frac{G_0(x,\varepsilon\phi(x))}{\varepsilon^2}\geq\frac{a_0(x)\phi^2(x)}{2}\quad \text{a.e. in } [\phi\neq 0]. \tag{1.117}$$

By integration and using relation (1.111), we obtain

$$\frac{1}{2}\int_{\Omega}|\nabla\phi|^2dx < \int_{\Omega}\frac{G_0(x,\varepsilon\phi(x))}{\varepsilon^2}, \tag{1.118}$$

provided $\varepsilon > 0$ is small enough. This concludes that $E(\varepsilon\phi) < 0$.

(c) Arguing by contradiction, there exists an unbounded sequence (u_n) in $H_0^1(\Omega)$ and a positive constant C such that for any $n \geq 1$, $E(u_n) \leq C$. Hence

$$\frac{1}{2}\int_{\Omega}|\nabla u_n|^2dx \leq \int_{\Omega}G_0(x,u_n)dx + C \leq C\Big(1+\|u_n\|^2_{L^2(\Omega)}\Big). \tag{1.119}$$

Set $u_n = t_n v_n$, where $t_n = \|u_n\|_{L^2(\Omega)} \to \infty$ and $v_n = u_n/\|u_n\|_{L^2(\Omega)}$. Then

$$\frac{1}{2}\int_{\Omega}|\nabla v_n|^2dx \leq C\left(1+\frac{1}{t_n^2}\right), \tag{1.120}$$

hence (v_n) is bounded in $H_0^1(\Omega)$. So, up to a subsequence, we can assume that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ in $L^2(\Omega)$ and a.e. in Ω. This implies that $\|v\|_{L^2(\Omega)} = 1$. We also deduce that

$$\frac{1}{2}\int_{\Omega}|\nabla v|^2dx \leq \liminf_{n\to\infty}\int_{\Omega}\frac{G_0(x,t_nv_n)}{t_n^2}dx \leq \int_{\Omega}\limsup_{n\to\infty}\frac{G_0(x,t_nv_n)}{t_n^2}dx. \tag{1.121}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega}\frac{G_0(x,t_nv_n)}{t_n^2}dx\\
&\quad=\int_{[v_n\leq 0]}\frac{G_0(x,t_nv_n)}{t_n^2}dx+\int_{[v_n>0]}\frac{G_0(x,t_nv_n)}{t_n^2}dx\\
&\quad=\int_{[v_n\leq 0]}\frac{G_0(x,t_nv_n)}{t_n^2}dx+\int_{[v\leq 0]}\frac{G_0(x,t_nv_n^+)}{t_n^2}dx+\int_{[v>0]}\frac{G_0(x,t_nv_n^+)}{t_n^2}dx\\
&\quad=I_1+I_2+I_3.
\end{aligned} \tag{1.122}$$

Trying to estimate these three integrals, we first observe that

$$I_1=\int_{[v_n\leq 0]}\frac{G_0(x,t_nv_n)}{t_n^2}dx \leq \frac{C}{t_n^2}\int_{[v_n\leq 0]}t_n|v_n|dx \leq \frac{C}{t_n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.123}$$

Next, we deduce that

$$I_2=\int_{[v\leq 0]}\frac{G_0(x,t_nv_n^+)}{t_n^2}dx \leq \frac{C}{t_n^2}\int_{[v\leq 0]}\left[t_n^2(v_n^+)^2+1\right]dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.124}$$

From the definition of a_∞ it follows that

$$\limsup_{v\to\infty}\frac{G_0(x,v)}{v^2} \leq \frac{a_\infty(x)}{2}, \tag{1.125}$$

hence

$$\limsup_{n\to\infty}\frac{G_0(x,t_nv_n^+)}{t_n^2} \leq \frac{a_\infty(x)v_n^2}{2} \quad \text{a.e. } x\in[v>0]. \tag{1.126}$$

By Fatou's lemma, we obtain

$$\limsup_{n\to\infty} I_3 \leq \frac{1}{2}\int_{[v>0]} a_\infty(x)v^2 dx. \tag{1.127}$$

By putting together these estimates, it follows that

$$\int_\Omega |\nabla v|^2 dx \leq \int_\Omega a_\infty(x)v^2 dx, \tag{1.128}$$

$$\int_\Omega |\nabla v|^2 dx - \int_\Omega a_\infty(x)v^2 dx \geq \alpha\|v\|^2_{L^2(\Omega)}, \tag{1.129}$$

where $\alpha := \lambda_1(-\Delta - a_\infty(x)) > 0$. Relations (1.128) and (1.129) imply that $v = 0$, which contradicts $\|v\|_{L^2(\Omega)} = 1$. □

We are now in position to prove the existence result stated in Theorem 1.17. Set $m := \inf_{u\in H^1_0(\Omega)} E(u)$. By Lemma 1.18, $-\infty < m < 0$. Let (u_n) be a minimizing sequence of E. Hence (u_n) is bounded in $H^1_0(\Omega)$. So, up to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H^1_0(\Omega)$, $u_n \to u$ in $L^2(\Omega)$ and a.e. in Ω. Thus, by Lemma 1.18, $E(u) \leq \liminf_{n\to\infty} E(u_n) = m$, hence $E(u) = m$. Since $E(u^+) \leq E(u)$, it follows that we can assume that $u \in H^1_0(\Omega)$ is a nonnegative solution of problem (1.95).

Next, we prove that $u \in L^\infty(\Omega)$. For any $k \geq 1$, set

$$g_k(x,u) = \begin{cases} \max\{g(x,u), -ku\} & \text{if } u > 0, \\ g(x,0) & \text{if } u \leq 0. \end{cases} \tag{1.130}$$

We observe that g_k satisfies conditions (1.89)–(1.91). Thus, there exists a unique solution u_k of the problem

$$\begin{gathered} -\Delta u_k = g_k(x,u_k) \quad \text{in } \Omega, \\ u_k \geq 0,\ u_k \not\equiv 0 \quad \text{in } \Omega, \\ u_k = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.131}$$

Then $u_k \in H^1_0(\Omega) \cap L^{2^\star}(\Omega)$ (if $N > 2$), hence $g_k(x,u_k) \in L^{2^\star}(\Omega)$. By Schauder elliptic estimates it follows that $u_k \subset W^{2,2^\star}(\Omega) \subset L^q(\Omega)$, where $q = 2^\star N/(N - 2^{1+2^\star})$, provided $N > 2^{1+2^\star}$; otherwise, $u_k \in L^\infty(\Omega)$. Since $2^\star = 2N/(N-2)$ (for any $N > 2$), we obtain $q = 2N/(N-6)$, provided $N > 6$. If $N \leq 5$ then $u_k \in L^\infty(\Omega)$. If $N > 6$ we continue the bootstrap technique and we deduce that $u_k \in L^r(\Omega)$ with $r = 2N/(N-10)$, provided $N > 10$ or $u_k \in L^\infty(\Omega)$, otherwise. Since N is fixed, we deduce that a finite number m of steps we obtain $N < 2m$ and this shows that $u_k \in L^\infty(\Omega)$.

Since u_k is a supersolution of problem (1.95) while 0 is a subsolution of this problem, we deduce that the (unique!) solution u of (1.95) satisfies $0 \leq u \leq u_k$, hence $u \in L^\infty(\Omega)$. At this stage we may argue that $u > 0$ in Ω. Indeed, since $u(x) \leq \|u\|_\infty$, our assumptions imply that

$$\frac{g(x,u(x))}{u(x)} \geq \frac{g(x,\|u\|_\infty)}{\|u\|_\infty} \geq -M, \quad \text{where } M > 0. \tag{1.132}$$

Thus, $g(x,u(x)) \geq -Mu(x)$ which implies that $-\Delta u + Mu \geq 0$ in Ω. Since $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$, the maximum principle implies that $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$.

It remains to argue that problem (1.95) has a unique solution. Let u_1 and u_2 be solutions of problem (1.95). We have already observed that $u_j \in L^\infty(\Omega)$, $u_j > 0$ in Ω, and $\partial u_j/\partial \nu < 0$ on $\partial\Omega$, for $j \in \{1,2\}$. Thus, by l'Hôpital's rule,

$$\frac{u_1}{u_2} \in L^\infty(\Omega), \qquad \frac{u_1^2}{u_2} \in H_0^1(\Omega). \tag{1.133}$$

Next, we multiply by $u_1^2 - u_2^2$ the equation satisfied by u_1. So, by Green's formula,

$$\begin{aligned}
\int_\Omega \frac{g(x,u_1)}{u_1}(u_1^2 - u_2^2)\,dx &= -\int_\Omega \Delta u_1\left(u_1 - \frac{u_2^2}{u_1}\right)dx \\
&= \int_\Omega \nabla u_1 \cdot \left(\nabla u_1 - \nabla\frac{u_2^2}{u_1}\right)dx \\
&= \int_\Omega |\nabla u_1|^2 dx - \int_\Omega \nabla u_1\left(\frac{2u_2}{u_1}\nabla u_2 - \frac{u_2^2}{u_1^2}\nabla u_1\right)dx \\
&= \int_\Omega \left(|\nabla u_1|^2 - \frac{2u_2}{u_1}\nabla u_1 \cdot \nabla u_2 + \frac{u_2^2}{u_1^2}|\nabla u_1|^2\right)dx.
\end{aligned} \tag{1.134}$$

After interchanging the solutions u_1 and u_2 and summing the two relations, we deduce that

$$\int_\Omega \left(\frac{g(x,u_1)}{u_1} - \frac{g(x,u_2)}{u_2}\right)(u_1^2 - u_2^2)\,dx = \int_\Omega \left(\left|\nabla u_1 - \frac{u_1}{u_2}\nabla u_2\right|^2 + \left|\nabla u_2 - \frac{u_2}{u_1}\nabla u_1\right|^2\right)dx. \tag{1.135}$$

Since the left-hand side is nonnegative, we deduce that $u_1 = u_2$ in Ω.

1.8. Applications to sublinear elliptic equations in anisotropic media

Brezis and Kamin are concerned in [32] with various questions related to the existence of bounded solutions of the sublinear elliptic equation *without condition at infinity*:

$$-\Delta u = \rho(x)u^\alpha \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{1.136}$$

where $0 < \alpha < 1$, $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$.

We summarize in what follows the main results obtained in Brezis and Kamin [32]. They proved that the *nonlinear* problem (1.136) has a bounded solution $u > 0$ if and only if the *linear* problem

$$-\Delta u = \rho(x) \quad \text{in } \mathbb{R}^N \tag{1.137}$$

has a bounded solution. In this case, problem (1.136) has a minimal positive solution and this solution satisfies $\liminf_{|x|\to\infty} u(x) = 0$. Moreover, the minimal solution is the unique positive solution of (1.136) which tends to zero at infinity. Brezis and Kamin also showed that if the potential $\rho(x)$ decays fast enough at infinity then problem (1.136) has a solution and, moreover, such a solution does not exist if $\rho(x)$ has a slow decay at infinity. For instance, if $\rho(x) = (1+|x|^p)^{-1}$, then (1.136) has a bounded solution if and only if $p > 2$. More generally, Brezis and Kamin proved that problem (1.136) has a bounded solution if and only if $\rho(x)$ is potentially bounded, that is, the mapping $x \mapsto \int_{\mathbb{R}^N} \rho(y)|x-y|^{2-N}dy \in L^\infty(\mathbb{R}^N)$. We refer to the monograph by Krasnoselskii [125] for various results on bounded domains for sublinear elliptic equations with zero Dirichlet boundary condition.

Our purpose in the first part of this section is to study the problem

$$\begin{gathered} -\Delta u = \rho(x) f(u) \quad \text{in } \mathbb{R}^N, \\ u > \ell \quad \text{in } \mathbb{R}^N, \\ u(x) \longrightarrow \ell \quad \text{as } |x| \longrightarrow \infty, \end{gathered} \tag{1.138}$$

where $N \geq 3$ and $\ell \geq 0$ is a real number.

Throughout the section we assume that the variable potential $\rho(x)$ satisfies $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$.

In our first result we suppose that the growth at infinity of the anisotropic potential $\rho(x)$ is given by

(ρ1) $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) := \max_{|x|=r} \rho(x)$.

Assumption (ρ1) has been first introduced in Naito [163].

The nonlinearity $f : (0,\infty) \to (0,\infty)$ satisfies $f \in C^{0,\alpha}_{\text{loc}}(0,\infty)$ $(0 < \alpha < 1)$ and has a sublinear growth, in the sense that

(f1) the mapping $u \mapsto f(u)/u$ is decreasing on $(0,\infty)$ and $\lim_{u\to\infty} f(u)/u = 0$.

We point out that condition (f1) does not require that f is smooth at the origin. The standard example of such a nonlinearity is $f(u) = u^p$, where $-\infty < p < 1$. We also observe that we study an equation of the same type as in Brezis and Kamin [32]. The main difference is that we require a certain asymptotic behavior at infinity of the solution.

Our main purpose in the present section is to consider both cases $\ell = 0$ and $\ell > 0$, under the natural assumption that the mapping $u \mapsto f(u)/u$ is decreasing on $(0,\infty)$. The results in this section are contained in the papers [45, 67].

Under the above hypotheses (ρ1) and (f1), our first result concerns the case $\ell > 0$.

Theorem 1.19. *Assume that $\ell > 0$. Then problem* (1.138) *has a unique classical solution.*

Next, consider the case $\ell = 0$. Instead of (ρ1) we impose the stronger condition

(ρ2) $\int_0^\infty r^{N-1}\Phi(r)dr < \infty$.

We remark that in Edelson [73] it is used the stronger assumption

$$\int_0^\infty r^{N-1+\lambda(N-2)}\Phi(r)dr < \infty, \quad \text{for some } \lambda \in (0,1). \tag{1.139}$$

Additionally, we suppose that

(f2) f is increasing in $(0,\infty)$ and $\lim_{u\searrow 0} f(u)/u = +\infty$.

A nonlinearity satisfying both (f1) and (f2) is $f(u) = u^p$, where $0 < p < 1$.

Our result in the case $\ell = 0$ is the following.

Theorem 1.20. *Assume that* $\ell = 0$ *and assumptions (ρ2), (f1), and (f2) are fulfilled. Then problem* (1.138) *has a unique classical solution.*

We point out that assumptions ($\rho 1$) and ($\rho 2$) are related to a celebrated class introduced by Kato, with wide and deep applications in potential theory and Brownian motion. We recall (see Aizenman and Simon [5]) that a real-valued measurable function ψ on $\mathbb{R}^N$ belongs to the Kato class $\mathcal{K}$ provided that

$$\lim_{\alpha \to 0} \sup_{x \in \mathbb{R}^N} \int_{|x-y| \leq \alpha} E(y) |\psi(y)| \, dy = 0, \tag{1.140}$$

where E denotes the fundamental solution of the Laplace equation. According to this definition and our assumption ($\rho 1$) (resp., ($\rho 2$)), it follows that $\psi = \psi(|x|) \in \mathcal{K}$, where $\psi(|x|) := |x|^{N-3}\Phi(|x|)$ (resp., $\psi(|x|) := |x|^{-1}\Phi(|x|)$), for all $x \neq 0$.

1.8.1. Entire solutions decaying to $\ell > 0$

In order to prove the existence of a solution to problem (1.138), we apply Theorem 1.17. For this purpose, we consider the boundary value problem

$$\begin{gathered} -\Delta u_k = \rho(x) f(u_k) \quad \text{if } |x| < k, \\ u_k > \ell \quad \text{if } |x| < k, n \\ u_k(x) = \ell \quad \text{if } |x| = k, \end{gathered} \tag{1.141}$$

where k is an arbitrary positive integer. Equivalently, the above Dirichlet problem can be rewritten:

$$\begin{gathered} -\Delta v_k = \rho(x) f(v_k + \ell) \quad \text{if } |x| < k, \\ v_k(x) = 0 \quad \text{if } |x| = k. \end{gathered} \tag{1.142}$$

In order to obtain a solution of problem (1.142), it is enough to check the hypotheses of Theorem 1.17.

(i) Since $f \in C(0,\infty)$ and $\ell > 0$, it follows that the mapping $v \mapsto \rho(x) f(v+\ell)$ is continuous in $[0,\infty)$.

(ii) From $\rho(x)(f(v+\ell)/v) = \rho(x)(f(v+\ell)/(v+\ell))((v+\ell)/v)$, using positivity of ρ and (f1) we deduce that the function $v \mapsto \rho(x)((v+\ell)/v)$ is decreasing on $(0,\infty)$.

(iii) For all $v \geq 0$, since $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, we obtain that $\rho \in L^\infty(B(0,k))$, so the condition (1.90) is satisfied.

(iv) By $\lim_{v\to\infty} f(v+\ell)/(v+1) = 0$ and $f \in C(0,\infty)$, there exists $M > 0$ such that $f(v+\ell) \leq M(v+1)$ for all $v \geq 0$. Therefore $\rho(x) f(v+\ell) \leq \|\rho\|_{L^\infty(B(0,k))} M(v+1)$ for all $v \geq 0$.

(v) We have

$$\begin{aligned} a_0(x) &= \lim_{v \searrow 0} \frac{\rho(x) f(v+\ell)}{v} = +\infty, \\ a_\infty(x) &= \lim_{v\to\infty} \frac{\rho(x) f(v+\ell)}{v} = \lim_{v\to\infty} \rho(x) \frac{f(v+\ell)}{v+\ell} \cdot \frac{v+\ell}{v} = 0. \end{aligned} \tag{1.143}$$

Thus, by Theorem 1.17, problem (1.142) has a unique solution v_k which, by the maximum principle, is positive in $|x| < k$. Then $u_k = v_k + \ell$ satisfies (1.141). Define $u_k = \ell$ for $|x| > k$. The maximum principle implies that $\ell \le u_k \le u_{k+1}$ in $\mathbb{R}^N$.

We now justify the existence of a continuous function $v : \mathbb{R}^N \to \mathbb{R}$, $v > \ell$, such that $u_k \le v$ in $\mathbb{R}^N$. We first construct a positive radially symmetric function w such that $-\Delta w = \Phi(r)$ $(r = |x|)$ in $\mathbb{R}^N$ and $\lim_{r\to\infty} w(r) = 0$. A straightforward computation shows that

$$w(r) = K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)d\sigma\, d\zeta, \tag{1.144}$$

where

$$K = \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)d\sigma\, d\zeta, \tag{1.145}$$

provided the integral is finite. An integration by parts yields

$$\begin{aligned}\int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)d\sigma\, d\zeta &= -\frac{1}{N-2}\int_0^r \frac{d}{d\zeta}\zeta^{2-N} \int_0^\zeta \sigma^{N-1}\Phi(\sigma)d\sigma\, d\zeta \\ &= \frac{1}{N-2}\Big(- r^{2-N}\int_0^r \sigma^{N-1}\Phi(\sigma)d\sigma + \int_0^r \zeta\Phi(\zeta)\Big) \\ &< \frac{1}{N-2}\int_0^\infty \zeta\Phi(\zeta) < +\infty.\end{aligned} \tag{1.146}$$

Moreover, w is decreasing and satisfies $0 < w(r) < K$ for all $r \ge 0$. Let $v > \ell$ be a function such that $w(r) = m^{-1}\int_0^{v(r)-\ell}(t/f(t+\ell))dt$, where $m > 0$ is chosen such that $Km \le \int_0^m (t/f(t+\ell))dt$.

Next, by L'Hôpital's rule for the case $\cdot/\infty$ (see [166, Theorem 3, page 319]) we have

$$\lim_{x\to\infty}\frac{\int_0^x (t/f(t+\ell))dt}{x} = \lim_{x\to\infty}\frac{x}{f(x+\ell)} = \lim_{x\to\infty}\frac{x+\ell}{f(x+\ell)}\cdot\frac{x}{x+\ell} = +\infty. \tag{1.147}$$

This means that there exists $x_1 > 0$ such that $\int_0^x (t/f(t)) \ge Kx$ for all $x \ge x_1$. It follows that for any $m \ge x_1$ we have $Km \le \int_0^m (t/f(t))dt$.

Since w is decreasing, we obtain that v is a decreasing function, too. Then

$$\int_0^{v(r)-\ell}\frac{t}{f(t+\ell)}dt \le \int_0^{v(0)-\ell}\frac{t}{f(t+\ell)}dt = mw(0) = mK \le \int_0^m \frac{t}{f(t+\ell)}dt. \tag{1.148}$$

It follows that $v(r) \le m + \ell$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to \ell$ as $r \to \infty$.

By the choice of v, we have

$$\nabla w = \frac{1}{m}\frac{v-\ell}{f(v)}\nabla v, \qquad \Delta w = \frac{1}{m}\frac{v-\ell}{f(v)}\Delta v + \frac{1}{m}\left(\frac{v-\ell}{f(v)}\right)'|\nabla v|^2. \tag{1.149}$$

Since the mapping $u \mapsto f(u)/u$ is decreasing on $(0,\infty)$, we deduce that

$$\Delta v < \frac{m}{v-\ell} f(v)\Delta w = -\frac{m}{v-\ell} f(v)\Phi(r) \leq -f(v)\Phi(r). \tag{1.150}$$

By (1.141), (1.150), and our hypothesis (f1), we obtain that $u_k(x) \leq v(x)$ for each $|x| \leq k$ and so, for all $x \in \mathbb{R}^N$.

In conclusion,

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v, \tag{1.151}$$

with $v(x) \to \ell$ as $|x| \to \infty$. Thus, there exists a function $u \leq v$ such that $u_k(x) \to u(x)$ as $k \to \infty$, for any $x \in \mathbb{R}^N$. In particular, this shows that $u > \ell$ in $\mathbb{R}^N$ and $u(x) \to \ell$ as $|x| \to \infty$.

A standard bootstrap argument shows that u is a classical solution of the problem (1.138).

Proof. To conclude the proof, it remains to show that the solution found above is unique. Suppose that u and v are solutions of (1.138). It is enough to show that $u \leq v$ or, equivalently, $\ln u(x) \leq \ln v(x)$, for any $x \in \mathbb{R}^N$. Arguing by contradiction, there exists $\overline{x} \in \mathbb{R}^N$ such that $u(\overline{x}) > v(\overline{x})$. Since $\lim_{|x|\to\infty}(\ln u(x) - \ln v(x)) = 0$, we deduce that $\max_{\mathbb{R}^N}(\ln u(x) - \ln v(x))$ exists and is positive. At this point, say x_0, we have

$$\nabla\big(\ln u(x_0) - \ln v(x_0)\big) = 0, \tag{1.152}$$

so

$$\frac{\nabla u(x_0)}{u(x_0)} = \frac{\nabla v(x_0)}{v(x_0)}. \tag{1.153}$$

By (f1), we obtain

$$\frac{f(u(x_0))}{u(x_0)} < \frac{f(v(x_0))}{v(x_0)}. \tag{1.154}$$

So, by (1.152) and (1.153),

$$\begin{aligned} 0 &\geq \Delta\big(\ln u(x_0) - \ln v(x_0)\big) \\ &= \frac{1}{u(x_0)}\cdot\Delta u(x_0) - \frac{1}{v(x_0)}\cdot\Delta v(x_0) - \frac{1}{u^2(x_0)}\cdot\big|\nabla u(x_0)\big|^2 + \frac{1}{v^2(x_0)}\cdot\big|\nabla v(x_0)\big|^2 \\ &= \frac{\Delta u(x_0)}{u(x_0)} - \frac{\Delta v(x_0)}{v(x_0)} = -\rho(x_0)\left(\frac{f(u(x_0))}{u(x_0)} - \frac{f(v(x_0))}{v(x_0)}\right) > 0, \end{aligned} \tag{1.155}$$

which is a contradiction. Hence $u \leq v$ and the proof is concluded. □

1.8.2. Entire solutions decaying to $\ell = 0$

Since f is an increasing positive function on $(0, \infty)$, there exists and is finite $\lim_{u \searrow 0} f(u)$, so f can be extended by continuity at the origin. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u_k &= \rho(x) f(u_k) \quad \text{if } |x| < k, \\ u_k(x) &= 0 \quad \text{if } |x| = k. \end{aligned} \tag{1.156}$$

Using the same arguments as in case $\ell > 0$, we deduce that conditions (1.89) and (1.90) are satisfied. In what concerns assumption (1.91), we use both assumptions (f1) and (f2). Hence $f(u) \le f(1)$ if $u \le 1$ and $f(u)/u \le f(1)$ if $u \ge 1$. Therefore, $f(u) \le f(1)(u + 1)$ for all $u \ge 0$, which proves (1.156). The existence of a solution for (1.97) follows from (1.97) and (1.98). These conditions are direct consequences of our assumptions $\lim_{u \to \infty} f(u)/u = 0$ and $\lim_{u \searrow 0} f(u)/u = +\infty$. Thus, by Theorem 1.17, problem (1.156) has a unique solution. Define $u_k(x) = 0$ for $|x| > k$. Using the same arguments as in case $\ell > 0$, we obtain $u_k \le u_{k+1}$ in $\mathbb{R}^N$.

Next, we prove the existence of a continuous function $v : \mathbb{R}^N \to \mathbb{R}$ such that $u_k \le v$ in $\mathbb{R}^N$. We first construct a positive radially symmetric function w satisfying $-\Delta w = \Phi(r)$ $(r = |x|)$ in $\mathbb{R}^N$ and $\lim_{r \to \infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma \, d\zeta, \tag{1.157}$$

where

$$K = \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma \, d\zeta, \tag{1.158}$$

provided the integral is finite. By integration by parts, we have

$$\begin{aligned} \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma \, d\zeta &= -\frac{1}{N-2} \int_0^r \frac{d}{d\zeta} \zeta^{2-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma \, d\zeta \\ &= \frac{1}{N-2} \left(- r^{2-N} \int_0^r \sigma^{N-1} \Phi(\sigma) d\sigma + \int_0^r \zeta \Phi(\zeta) d\zeta \right) \\ &< \frac{1}{N-2} \int_0^\infty \zeta \Phi(\zeta) d\zeta < \infty. \end{aligned} \tag{1.159}$$

Therefore,

$$w(r) < \frac{1}{N-2} \cdot \int_0^\infty \zeta \Phi(\zeta) d\zeta \quad \forall r > 0. \tag{1.160}$$

Let v be a positive function such that $w(r) = c^{-1} \int_0^{v(r)} t/f(t) dt$, where $c > 0$ is chosen such that $Kc \le \int_0^c t/f(t) dt$. We argue in what follows that we can find $c > 0$ with this property. Indeed, by L'Hôpital's rule,

$$\lim_{x \to \infty} \frac{\int_0^x (t/f(t)) dt}{x} = \lim_{x \to \infty} \frac{x}{f(x)} = +\infty. \tag{1.161}$$

This means that there exists $x_1 > 0$ such that $\int_0^x t/f(t)dt \geq Kx$ for all $x \geq x_1$. It follows that for any $c \geq x_1$ we have $Kc \leq \int_0^c t/f(t)dt$.

On the other hand, since w is decreasing, we deduce that v is a decreasing function, too. Hence

$$\int_0^{v(r)} \frac{t}{f(t)}dt \leq \int_0^{v(0)} \frac{t}{f(t)}dt = c \cdot w(0) = c \cdot K \leq \int_0^c \frac{t}{f(t)}dt. \tag{1.162}$$

It follows that $v(r) \leq c$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to 0$ as $r \to \infty$.

By the choice of v we have

$$\nabla w = \frac{1}{c} \cdot \frac{v}{f(v)} \nabla v, \qquad \Delta w = \frac{1}{c}\frac{v}{f(v)}\Delta v + \frac{1}{c}\left(\frac{v}{f(v)}\right)' |\nabla v|^2. \tag{1.163}$$

Combining the fact that $f(u)/u$ is a decreasing function on $(0, \infty)$ with relation (1.163), we deduce that

$$\Delta v < c\frac{f(v)}{v}\Delta w = -c\frac{f(v)}{v}\Phi(r) \leq -f(v)\Phi(r). \tag{1.164}$$

By (1.156) and (1.164) and using our hypothesis (f2), as already done for proving the uniqueness in the case $\ell > 0$, we obtain that $u_k(x) \leq v(x)$ for each $|x| \leq k$ and so, for all $x \in \mathbb{R}^N$.

We have obtained a bounded increasing sequence

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v, \tag{1.165}$$

with v vanishing at infinity. Thus, there exists a function $u \leq v$ such that $u_k(x) \to u(x)$ as $k \to \infty$, for any $x \in \mathbb{R}^N$. A standard bootstrap argument implies that u is a classical solution of the problem (1.138).

We argue in what follows that the solution found above is unique. For this purpose we split the proof into two steps. Assume that u_1 and u_2 are solutions of problem (1.138). We first prove that if $u_1 \leq u_2$ then $u_1 = u_2$ in $\mathbb{R}^N$. In the second step, we find a positive solution $u \leq \min\{u_1, u_2\}$ and thus, using the first step, we deduce that $u = u_1$ and $u = u_2$, which proves the uniqueness.

Step 1. We show that $u_1 \leq u_2$ in $\mathbb{R}^N$ implies $u_1 = u_2$ in $\mathbb{R}^N$. Indeed, since

$$u_1\Delta u_2 - u_2\Delta u_1 = \rho(x)u_1u_2\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right) \geq 0, \tag{1.166}$$

it is sufficient to check that

$$\int_{\mathbb{R}^N} (u_1\Delta u_2 - u_2\Delta u_1) = 0. \tag{1.167}$$

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be such that $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$, and denote $\psi_n := \psi(x/n)$ for any positive integer n. Set

$$I_n := \int_{\mathbb{R}^N} (u_1\Delta u_2 - u_2\Delta u_1)\psi_n \, dx. \tag{1.168}$$

We claim that $I_n \to 0$ as $n \to \infty$. Indeed,

$$|I_n| \leq \int_{\mathbb{R}^N} |u_1 \Delta u_2| \psi_n \, dx + \int_{\mathbb{R}^N} |u_2 \Delta u_1| \psi_n \, dx. \tag{1.169}$$

So, by symmetry, it is enough to prove that $J_n := \int_{\mathbb{R}^N} |u_1 \Delta u_2| \psi_n dx \to 0$ as $n \to \infty$. But, from (1.138),

$$\begin{aligned} J_n &= \int_{\mathbb{R}^N} |u_1 f(u_2)\rho(x)| \psi_n dx \\ &= \int_n^{2n} \int_{|x|=r} |u_1(x) f(u_2(x))\rho(x)| \, dx \, dr \\ &\leq \int_n^{2n} \Phi(r) \int_{|x|=r} |u_1(x) f(u_2(x))| \, dx \, dr \\ &\leq \int_n^{2n} \Phi(r) \int_{|x|=r} |u_1(x)| M(u_2+1) \, dx \, dr. \end{aligned} \tag{1.170}$$

Since $u_1(x)$, $u_2(x) \to 0$ as $|x| \to \infty$, we deduce that u_1 and u_2 are bounded in $\mathbb{R}^N$. Returning to (1.170), we have

$$\begin{aligned} J_n &\leq M(\|u_2\|_{L^\infty(\mathbb{R}^N)} + 1) \sup_{|x|\geq n} |u_1(x)| \cdot \frac{\omega_N}{N} \int_n^{2n} \Phi(r) r^{N-1} dr \\ &\leq C \int_0^\infty \Phi(r) r^{N-1} dr \cdot \sup_{|x|\geq n} |u_1(x)|. \end{aligned} \tag{1.171}$$

Since $u_1(x) \to 0$ as $|x| \to \infty$, we have $\sup_{|x|\geq n} |u_1(x)| \to 0$ as $n \to \infty$ which shows that $J_n \to 0$. In particular, this implies $I_n \to 0$ as $n \to \infty$.

Taking $f_n := (u_1 \Delta u_2 - u_2 \Delta u_1)\psi_n$, we deduce $f_n(x) \to u_1(x)\Delta u_2(x) - u_2(x)\Delta u_1(x)$ as $n \to \infty$. To apply Lebesgue's dominated convergence theorem, we need to show that $u_1 \Delta u_2 - u_2 \Delta u_1 \in L^1(\mathbb{R}^N)$. For this purpose it is sufficient to prove that $u_1 \Delta u_2 \in L^1(\mathbb{R}^N)$. Indeed,

$$\int_{\mathbb{R}^N} |u_1 \Delta u_2| \leq \|u_1\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\Delta u_2| = C \int_{\mathbb{R}^N} |\rho(x) f(u_2)|. \tag{1.172}$$

Thus, using $f(u) \leq f(1)(u+1)$ and since u_2 is bounded, the above inequality yields

$$\begin{aligned} \int_{\mathbb{R}^N} |u_1 \Delta u_2| &\leq C \int_{\mathbb{R}^N} |\rho(x)(u_2+1)| \\ &\leq C \int_0^\infty \int_{|x|=r} \Phi(r) \, dx \, dr \\ &\leq C \int_0^\infty \Phi(r) r^{N-1} < +\infty. \end{aligned} \tag{1.173}$$

This shows that $u_1 \Delta u_2 \in L^1(\mathbb{R}^N)$ and the proof of Step 1 is completed.

Step 2. Let u_1, u_2 be arbitrary solutions of problem (1.138). For all integer $k \geq 1$, denote $\Omega_k := \{x \in \mathbb{R}^N;\ |x| < k\}$. Theorem 1.17 implies that the problem

$$\begin{aligned} -\Delta v_k &= \rho(x) f(v_k) \quad \text{in } \Omega_k, \\ v_k &= 0 \quad \text{on } \partial\Omega_k \end{aligned} \tag{1.174}$$

has a unique solution $v_k \geq 0$. Moreover, by the maximum principle, $v_k > 0$ in Ω_k. We define $v_k = 0$ for $|x| > k$. Applying again the maximum principle we deduce that $v_k \leq v_{k+1}$ in $\mathbb{R}^N$. Now we prove that $v_k \leq u_1$ in $\mathbb{R}^N$, for all $k \geq 1$. Obviously, this happens outside Ω_k. On the other hand,

$$\begin{aligned} -\Delta u_1 &= \rho(x) f(u_1) \quad \text{in } \Omega_k, \\ u_1 &> 0 \quad \text{on } \partial\Omega_k, \end{aligned} \tag{1.175}$$

Arguing by contradiction, we assume that there exists $\overline{x} \in \Omega_k$ such that $v_k(\overline{x}) > u_1(\overline{x})$. Consider the function $h : \Omega_k \to \mathbb{R}$, $h(x) = \ln v_k(x) - \ln u(x)$. Since u_1 is bounded in Ω_k and $\inf_{\partial\Omega_k} u_1 > 0$, we have $\lim_{|x|\to k} h(x) = -\infty$. We deduce that $\max_{\Omega_k}(\ln v_k(x) - \ln u_1(x))$ exists and is positive. Using the same argument as in the case $\ell > 0$ we deduce that $v_k \leq u_1$ in Ω_k, so in $\mathbb{R}^N$. Similarly, we obtain $v_k \leq u_2$ in $\mathbb{R}^N$. Hence $v_k \leq \overline{u} := \min\{u_1, u_2\}$. Therefore $v_k \leq v_{k+1} \leq \cdots \leq \overline{u}$. Thus, there exists a function u such that $v_k \to u$ a.e. in $\mathbb{R}^N$. Repeating a previous argument we deduce that $u \leq \overline{u}$ is a classical solution of problem (1.138). Moreover, since $u \geq v_k > 0$ in Ω_k and for all $k \geq 1$, we deduce that $u > 0$ in $\mathbb{R}^N$. This concludes the proof of Step 2.

Combining Steps 1 and 2, we conclude that $u_1 = u_2$ in $\mathbb{R}^N$.

1.9. Bibliographical notes

Theorem 1.2 goes back to Scorza Dragoni [209] who considered the solvability of Sturm-Liouville problems of the type

$$\begin{aligned} y''(x) + f(x, y(x), y'(x)) &= 0 \quad \text{in } (a, b), \\ x(a) = A, \qquad x(b) &= B, \end{aligned} \tag{1.176}$$

by using comparison principles. The concepts of lower and upper solutions were introduced by Nagumo [162] in 1937 who proved, using also the shooting method, the existence of at least one solution for problem (1.176).

The Krasnoselski theorem is a major tool for proving the uniqueness of the solution for many classes of nonlinear elliptic problems. The Brezis-Oswald theorem formulates a necessary and sufficient condition for the existence of a solution in the case of nonlinearities having a sublinear growth. In the same setting Theorem 1.17 establishes the uniqueness of the solution in the case of sublinear problems on bounded domains. This result is extended in Theorems 1.19 and 1.20 in the case of sublinear problems with variable potential on the whole space. Brezis and Kamin studied in [32] the existence of bounded solutions of sublinear elliptic equations *without condition at infinity*. We deal with entire solutions decaying either to 0 or to $\ell > 0$ at infinity. We also point out that the qualitative analysis we develop is strictly related to a class of variable potentials belonging to a certain Kato class.

2 Bifurcation problems

Ce que l'on désire ardemment, constamment, on l'obtient toujours.

Napoléon Bonaparte

2.1. Introduction

Bifurcation problems have a long history and their treatment goes back to the 18th century. One of the first bifurcation problems is related to the buckling of a thin rod under thrust and was investigated by Daniel Bernoulli and Euler around 1744. In the case in which the rod is free to rotate at both end points, this yields the one-dimensional bifurcation problem:

$$\begin{aligned} u'' + \lambda \sin u = 0 \quad &\text{in } (0, L),\\ 0 \le u \le \pi&,\\ u'(0) = u'(L) = 0&. \end{aligned} \tag{2.1}$$

The role of bifurcation problems in applied mathematics has been synthesized by Kielhöfer [120], who observed that "the buckling of the Euler rod, the appearance of Taylor vortices, and the onset of oscillations in an electric circuit, for instance, all have a common cause: a specific physical parameter crosses a threshold, and that event forces the system to the organization of a new state that differs considerably from that observed before."

In this chapter, we develop some basic methods for the qualitative analysis of bifurcation problems. We are mainly concerned with the implicit function theorem, whose germs appear in the writings of Newton, Leibniz, and Lagrange. In its simplest form, this theorem involves solving an equation of the form

$$F(u, \lambda) = 0, \qquad F(0, 0) = 0 \tag{2.2}$$

for a function u, where λ is a small parameter. The basic assumption is that $F_u(u, \lambda)$ is invertible for u and λ "close" to their respective origins, but with some loss of derivatives when computing the inverse. In his celebrated proof of the result that any compact Riemannian manifold may be isometrically embedded in some Euclidean space, Nash

[165] developed a deep extension of the implicit function theorem. His ideas have been extended by various people to a technique which is now called the Nash-Moser theory. We point out that John F. Nash Jr. is an outstanding scientist with deep contributions in many domains. He received the 1994 Nobel Prize in Economics for his "Pioneering analysis of equilibria in the theory of non-cooperative games."

2.2. Abstract theorems

Let X, Y be Banach spaces. Our aim is to develop a general method which will enable us to solve equations of the type

$$F(u,\lambda) = v, \tag{2.3}$$

where $F : X \times \mathbb{R} \to Y$ is a prescribed sufficiently smooth function and $v \in Y$ is given.

Theorem 2.1. *Let X, Y be real Banach spaces and let $(u_0,\lambda_0) \in X \times \mathbb{R}$. Consider a C^1-mapping $F = F(u,\lambda) : X \times \mathbb{R} \to Y$ such that the following conditions hold:*

(i) *$F(u_0,\lambda_0) = 0$;*
(ii) *the linear mapping $F_u(u_0,\lambda_0) : X \to Y$ is bijective.*

Then there exists a neighborhood U_0 of u_0 and a neighborhood V_0 of λ_0 such that for every $\lambda \in V_0$ there is a unique element $u(\lambda) \in U_0$ so that $F(u(\lambda),\lambda) = 0$.

Moreover, the mapping $V_0 \ni \lambda \mapsto u(\lambda)$ is of class C^1.

Proof. Consider the mapping $\Phi(u,\lambda) : X \times \mathbb{R} \to Y \times \mathbb{R}$ defined by $\Phi(u,\lambda) = (F(u,\lambda),\lambda)$. It is obvious that $\Phi \in C^1$. We apply to Φ the inverse function theorem. For this aim, it remains to verify that the mapping $\Phi'(u_0,\lambda_0) : X \times \mathbb{R} \to Y \times \mathbb{R}$ is bijective. Indeed, we have

$$\begin{aligned}
&\Phi(u_0 + tu,\lambda_0 + t\lambda) \\
&\quad = \big(F(u_0 + tu,\lambda_0 + t\lambda),\lambda_0 + t\lambda\big) \\
&\quad = \big(F(u_0,\lambda_0) + F_u(u_0,\lambda_0)\cdot(tu) + F_\lambda(u_0,\lambda_0)\cdot(t\lambda) + o(1),\lambda_0 + t\lambda\big).
\end{aligned} \tag{2.4}$$

It follows that

$$F'(u_0,\lambda_0) = \begin{pmatrix} F_u(u_0,\lambda_0) & F_\lambda(u_0,\lambda_0) \\ 0 & I \end{pmatrix} \tag{2.5}$$

which is a bijective operator, by our hypotheses. Thus, by the inverse function theorem, there exists a neighborhood $\mathcal{U}$ of the point (u_0,λ_0) and a neighborhood $\mathcal{V}$ of $(0,\lambda)$ such that the equation

$$\Phi(u,\lambda) = (f,\lambda_0) \tag{2.6}$$

has a unique solution, for every $(f,\lambda) \in \mathcal{V}$. Now it is sufficient to take here $f = 0$ and our conclusion follows. □

With a similar proof one can justify the following *global* version of the implicit function theorem.

Theorem 2.2. *Assume* $F : X \times \mathbb{R} \to Y$ *is a* C^1*-function on* $X \times \mathbb{R}$ *satisfying*

(i) $F(0,0) = 0$,
(ii) *the linear mapping* $F_u(0,0) : X \to Y$ *is bijective.*

Then there exist an open neighborhood I *of 0 and a* C^1 *mapping* $I \ni \lambda \mapsto u(\lambda)$ *such that* $u(0) = 0$ *and* $F(u(\lambda),\lambda) = 0$.

The following result will be of particular importance in the next applications.

Theorem 2.3. *Assume the same hypotheses on* F *as in Theorem 2.2. Then there exists an open and maximal interval* I *containing the origin and there exists a unique* C^1*-mapping* $I \ni \lambda \mapsto u(\lambda)$ *such that the following hold:*

(a) $F(u(\lambda),\lambda) = 0$, *for every* $\lambda \in I$;
(b) *the linear mapping* $F_u(u(\lambda),\lambda)$ *is bijective, for any* $\lambda \in I$;
(c) $u(0) = 0$.

Proof. Let u_1, u_2 be solutions and consider the corresponding open intervals I_1 and I_2 on which these solutions exist, respectively. It follows that $u_1(0) = u_2(0) = 0$ and

$$\begin{aligned} F(u_1(\lambda),\lambda) &= 0, \quad \text{for every } \lambda \in I_1, \\ F(u_2(\lambda),\lambda) &= 0, \quad \text{for every } \lambda \in I_2. \end{aligned} \tag{2.7}$$

Moreover, the mappings $F_u(u_1(\lambda),\lambda)$ and $F_u(u_2(\lambda),\lambda)$ are one-to-one and onto on I_1, respectively. I_2. But, for λ sufficiently close to 0 we have $u_1(\lambda) = u_2(\lambda)$. We wish to show that we have global uniqueness. For this aim, let

$$I = \{\lambda \in I_1 \cap I_2;\ u_1(\lambda) = u_2(\lambda)\}. \tag{2.8}$$

Our aim is to show that $I = I_1 \cap I_2$. We first observe that $0 \in I$, so $I \neq \emptyset$. A standard argument then shows that I is closed in $I_1 \cap I_2$. In order to show that $I = I_1 \cap I_2$, it is sufficient now to prove that I is an open set in $I_1 \cap I_2$. The proof of this statement follows by applying Theorem 2.1 for λ instead of 0. Thus, $I = I_1 \cap I_2$.

Now, in order to justify the existence of a maximal interval I, we consider the C^1-curves $u_n(\lambda)$ defined on the corresponding open intervals I_n, such that $0 \in I_n$, $u_n(0) = u_0$, $F(u_n(\lambda),\lambda) = 0$ and $F_u(u_n(\lambda),\lambda)$ is an isomorphism, for any $\lambda \in I_n$. Now a standard argument enables us to construct a maximal solution on the set $\cup_n I_n$. □

Corollary 2.4. *Let* X, Y *be Banach spaces and let* $F : X \to Y$ *be a* C^1*-function. Assume that the linear mapping* $F_u(u) : X \to Y$ *is bijective, for every* $u \in X$ *and there exists* $C > 0$ *such that* $\|(F_u(u))^{-1}\| \leq C$, *for any* $u \in X$. *Then* F *is onto.*

Proof. Assume, without loss of generality, that $F(0) = 0$ and fix arbitrarily $f \in Y$. Consider the operator $F(u,\lambda) = F(u) - \lambda f$, defined on $X \times \mathbb{R}$. Then, by Theorem 2.3, there exists a C^1-function $u(\lambda)$ which is defined on a maximal interval I such that $F(u(\lambda)) = \lambda f$. In particular, $u := u(1)$ is a solution of the equation $F(u) = f$. We assert that $I = \mathbb{R}$. Indeed, we have

$$u_\lambda(\lambda) = \left(F_u(u)\right)^{-1} f, \tag{2.9}$$

so u is a Lipschitz map on I, which implies $I = \mathbb{R}$.

The implicit function theorem is used to solve equations of the type $F(u) = f$, where $F \in C^1(X, Y)$. A simple method for proving that F is onto, that is, $\operatorname{Im} F = Y$ is to prove the following:

(i) $\operatorname{Im} F$ is open;
(ii) $\operatorname{Im} F$ is closed.

For showing (i), usually we use the inverse function theorem, more exactly, if $F_u(u)$ is one-to-one, for every $u \in X$, then (i) holds. A sufficient condition for that (ii) holds is that F is a proper map. □

Another variant of the implicit function theorem is stated in the following result.

Theorem 2.5. *Let $F(u,\lambda)$ be a C^1-mapping in a neighborhood of $(0,0)$ and such that $F(0,0) = 0$. Assume that*

(i) $\operatorname{Im} F_u(0,0) = Y$;
(ii) *the space $X_1 := \operatorname{Ker} F_u(0,0)$ has a closed complement X_2.*

Then there exist $B_1 = \{u_1 \in X; \|u_1\| < \delta\}$, $B_2 = \{\lambda \in \mathbb{R}; |\lambda| < r\}$, $B_3 = \{g \in Y; \|g\| < R\}$, and a neighborhood U of the origin in X_2 such that, for any $u_1 \in B_1$, $\lambda \in B_2$, and $g \in B_3$, there exists a unique solution $u_2 = \varphi(u_1,\lambda,g) \in U$ of the equation

$$F\left(u_1 + \varphi(u_1,\lambda,g),\lambda\right) = g. \tag{2.10}$$

Proof. Let $\Gamma = X \times \mathbb{R} \times Y$, that is, every element $v \in \Gamma$ has the form $v = (u_1,\lambda,g)$. It remains to apply then implicit function theorem to the mapping $G : X \times \Gamma \to Y$ which is defined by $G(u_2,v) = F(u_1 + u_2,\lambda) - g$. □

We conclude this section with the following elementary example: let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let g be a C^1 real function defined on a neighborhood of 0 and such that $g(0) = 0$. Consider the problem

$$\begin{gathered} -\Delta u = g(u) + f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.11}$$

Assume that $g'(0)$ is not a real number of $-\Delta$ in $H_0^1(\Omega)$, say $g'(0) \leq 0$. If f is sufficiently small, then the problem (2.11) has a unique solution, by the implicit function theorem. Indeed, it is enough to apply Theorem 2.1 to the operator $F(u) = -\Delta u - g(u)$ and after observing that $F_u(0) = -\Delta - g'(0)$. There are at least two distinct possibilities for defining F:

(i) $F : C_0^{2,\alpha}(\overline{\Omega}) \to C_0^{\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, or

(ii) $F : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$; In order to obtain classical solutions (by a standard bootstrap argument that we will describe later), it is sufficient to choose $p > N/2$.

2.3. A basic bifurcation theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 mapping which is convex, positive and such that $f'(0) > 0$.

Consider the problem

$$
\begin{aligned}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{2.12}
$$

where λ is a positive parameter. We are looking for classical solutions of this problem, that is, $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Trying to apply the implicit function theorem to our problem (2.12), set

$$
X = \{u \in C^{2,\alpha}(\overline{\Omega}); u = 0 \text{ on } \partial\Omega\}
\tag{2.13}
$$

and $Y = C^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Define $F(u,\lambda) = -\Delta u - \lambda f(u)$. It is clear that F verifies all the assumptions of the implicit function theorem. Hence, there exist a maximal neighborhood of the origin I and a unique map $u = u(\lambda)$ which is solution of the problem (P) and such that the linearized operator $-\Delta - \lambda f'(u(\lambda))$ is bijective. In other words, for every $\lambda \in I$, problem (2.12) admits a stable solution which is given by the implicit function theorem. Let $\lambda^\star := \sup I \le +\infty$. We will denote from now on by $\lambda_1(-\Delta - a)$ the first eigenvalue in $H_0^1(\Omega)$ of the operator $-\Delta - a$, where $a \in L^\infty(\Omega)$.

Our main purpose in this section is to prove the following basic bifurcation theorem. To the best of our knowledge, this result is due to Keller and Cohen [119]; we also refer to Amann [9] for applications to nonlinear partial differential equations.

Theorem 2.6. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^2 which is convex, positive and such that $f'(0) > 0$. Then the following hold:*

(i) $\lambda^\star < +\infty$;

(ii) $\lambda_1(-\Delta - \lambda f'(u(\lambda))) > 0$;

(iii) *the mapping $I \ni \lambda \mapsto u(x,\lambda)$ is increasing, for every $x \in \Omega$;*

(iv) *for every $\lambda \in I$ and $x \in \Omega$, one has $u(x,\lambda) > 0$;*

(v) *there is no solution of problem* (2.12), *provided that $\lambda > \lambda^\star$;*

(vi) *$u(\lambda)$ is a minimal solution of problem* (2.12)*;*

(vii) *$u(\lambda)$ is the unique stable solution of the problem* (2.12).

Proof. (i) It is obvious, by the variational characterization of the first eigenvalue, that if $a, b \in L^\infty(\Omega)$ and $a \le b$ then

$$
\lambda_1(-\Delta - a(x)) \ge \lambda_1(-\Delta - b(x)).
\tag{2.14}
$$

Assume (iv) is already proved. Thus, by the convexity of f, $f'(u(\lambda)) \geq f'(0) > 0$, which implies that

$$\lambda_1(-\Delta - \lambda f'(0)) \geq \lambda_1(-\Delta - \lambda f'(u(\lambda))) > 0, \tag{2.15}$$

for every $\lambda \in I$. This implies $\lambda_1 - \lambda f'(0) > 0$, for any $\lambda < \lambda^\star$, that is, $\lambda^\star \leq \lambda_1/f'(0) < +\infty$.

(ii) Set $\varphi(\lambda) = \lambda_1(-\Delta - \lambda f'(u(\lambda)))$. So, $\varphi(0) = \lambda_1 > 0$ and, for every $\lambda < \lambda^\star$, $\varphi(\lambda) \neq 0$, since the linearized operator $-\Delta - \lambda f'(u(\lambda))$ is bijective, by the implicit function theorem. Now, by the continuity of the mapping $\lambda \mapsto \lambda f'(u(\lambda))$, it follows that φ is continuous, which implies, by the above remarks, $\varphi > 0$ on $[0, \lambda^\star)$.

(iii) We differentiate in (2.12) with respect to λ. Thus

$$-\Delta u_\lambda = f(u(\lambda)) + \lambda f'(u(\lambda)) \cdot u_\lambda \quad \text{in } \Omega \tag{2.16}$$

and $u_\lambda = 0$ on $\partial\Omega$. Hence

$$(-\Delta - \lambda f'(u(\lambda)))u_\lambda = f(u(\lambda)) \quad \text{in } \Omega. \tag{2.17}$$

But the operator $(-\Delta - \lambda f'(u(\lambda)))$ is coercive. So, by Stampacchia's maximum principle, either $u_\lambda \equiv 0$ in Ω, or $u_\lambda > 0$ in Ω. The first variant is not convenient, since it would imply that $f(u(\lambda)) = 0$, which is impossible, by our initial hypotheses. It remains that $u_\lambda > 0$ in Ω.

(iv) follows from (iii).

(v) Assume that there exists some $v > \lambda^\star$ and there exists a corresponding solution v to our problem (2.12). □

Claim 3. $u(\lambda) < v$ in Ω, for every $\lambda < \lambda^\star$.

Proof of Claim 3. By the convexity of f it follows that

$$-\Delta(v - u(\lambda)) = vf(v) - \lambda f(u(\lambda)) \geq \lambda(f(v) - f(u(\lambda))) \geq \lambda f'(u(\lambda))(v - u(\lambda)). \tag{2.18}$$

Hence

$$-\Delta(v - u(\lambda)) - \lambda f'(u(\lambda))(v - u(\lambda)) = (-\Delta - \lambda f'(u(\lambda)))(v - u(\lambda)) \geq 0 \quad \text{in } \Omega, \tag{2.19}$$

since the operator $-\Delta - \lambda f'(u(\lambda))$ is coercive. Thus, by Stampacchia's maximum principle, $v \geq u(\lambda)$ in Ω, for every $\lambda < \lambda^\star$.

Hence, $u(\lambda)$ is bounded in L^∞ by v. Passing to the limit as $\lambda \to \lambda^\star$ we find that $u(\lambda) \to u^\star < +\infty$ and $u^\star = 0$ on $\partial\Omega$.

We prove in what follows that

$$\lambda_1(-\Delta - \lambda^\star f'(u^\star)) = 0. \tag{2.20}$$

We already know that $\lambda_1(-\Delta - \lambda^\star f'(u^\star)) \geq 0$. Assume that $\lambda_1(-\Delta - \lambda^\star f'(u^\star)) > 0$, so, this operator is coercive. We apply the implicit function theorem to $F(u, \lambda) = -\Delta u - \lambda f(u)$ at the point $(u^\star, \lambda^\star)$. We obtain that there is a curve of solutions of problem (2.12) passing through $(u^\star, \lambda^\star)$, which contradicts the maximality of $\lambda^\star$. We have obtained that $\lambda_1(-\Delta - \lambda^\star f'(u^\star)) = 0$. So, there exists $\varphi_1 > 0$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, so that

$$-\Delta\varphi_1 - \lambda^\star f'(u^\star)\varphi_1 = 0 \quad \text{in } \Omega. \tag{2.21}$$

Passing to the limit as $\lambda \to \lambda^\star$ in the relation

$$\big(-\Delta - \lambda f'(u(\lambda))\big)(v - u(\lambda)) \geq 0, \tag{2.22}$$

we find

$$\big(-\Delta - \lambda^\star f'(u^\star)\big)(v - u^\star) \geq 0. \tag{2.23}$$

Multiplying this inequality by φ_1 and integrating, we obtain

$$-\int_\Omega (v - u^\star)\Delta\varphi_1 dx - \lambda^\star \int_\Omega f'(u^\star)(v - u^\star)\varphi_1 dx \geq 0. \tag{2.24}$$

In fact, by (2.21), the above relation is an equality, which implies that

$$-\Delta(v - u^\star) = \lambda^\star f'(u^\star)(v - u^\star) \quad \text{in } \Omega. \tag{2.25}$$

It follows that $vf(v) = \lambda^\star f(u^\star)$ in Ω. But $v > \lambda^\star$ and $f(v) \geq f(u^\star)$. So, $f(v) = 0$ which is impossible.

(vi) Fix an arbitrary $\lambda < \lambda^\star$. Assume that v is another solution of problem (2.12). We have

$$-\Delta(v - u(\lambda)) = \lambda f(v) - \lambda f(u(\lambda)) \geq \lambda f'(u(\lambda))(v - u(\lambda)) \quad \text{in } \Omega. \tag{2.26}$$

Again, by Stampacchia's maximum principle applied to the coercive operator $-\Delta - \lambda f'(u(\lambda))$, we find that $v \geq u(\lambda)$ in Ω.

(vii) Let v be another stable solution, for some $\lambda < \lambda^\star$. With the same reasoning as in (vi), but applied to the coercive operator $-\Delta - \lambda f'(v)$, we get that $u(\lambda) \geq v$. Finally, $u(\lambda) = v$. □

2.4. Qualitative properties of the minimal solution near the bifurcation point

We assume, throughout this section, that

$$\lim_{t\to\infty} \frac{f(t)}{t} = a > 0. \tag{2.27}$$

We impose in this section a supplementary hypothesis to f, which essentially means that as $t \to \infty$, then f increases faster than t^a, for some $a > 1$. In this case, we will prove that problem (2.12) has a weak solution $u^* \in H_0^1(\Omega)$ provided $\lambda = \lambda^*$. However, we cannot obtain in this case a supplementary regularity of u^*.

Theorem 2.7. *Assume that f satisfies the following additional condition: there is some $a > 0$ and $\mu > 1$ such that, for every $t \geq a$,*

$$tf'(t) \geq \mu f(t). \tag{2.28}$$

The following hold

(i) *problem (2.12) has a solution u^*, provided that $\lambda = \lambda^*$;*
(ii) *u^* is the weak limit in $H_0^1(\Omega)$ of stable solutions $u(\lambda)$, if $\lambda \nearrow \lambda^*$;*
(iii) *for every $v \in H_0^1(\Omega)$,*

$$f'(u^*)v^2 \in L^1(\Omega), \lambda^* \int_\Omega f'(u^*)v^2 dx \leq \int_\Omega |\nabla v|^2 dx. \tag{2.29}$$

Proof. For every $v \in H_0^1(\Omega)$ and $\lambda \in [0, \lambda^*)$ we have, by Theorem 2.6,

$$\lambda^* \int_\Omega f'(u(\lambda))v^2 dx \leq \int_\Omega |\nabla v|^2 dx. \tag{2.30}$$

Choosing here $v = u(\lambda)$, we find

$$\lambda^* \int_\Omega f'(u(\lambda))u^2(\lambda)dx \leq \int_\Omega |\nabla u(\lambda)|^2 dx = \lambda \int_\Omega f(u(\lambda))u(\lambda)dx. \tag{2.31}$$

So, if for a as in our hypothesis we define

$$\Omega(\lambda) = \{x \in \Omega;\ u(\lambda)(x) > a\}, \tag{2.32}$$

then

$$\begin{aligned} \int_\Omega f'(u(\lambda))u^2(\lambda)dx &\leq \frac{\lambda}{\lambda^*} \int_\Omega f(u(\lambda))u(\lambda)dx \\ &\leq \frac{\lambda}{\lambda^*} \Big[\int_{\Omega(\lambda)} f(u(\lambda))u(\lambda)dx + a \cdot |\Omega| \cdot f(a) \Big]. \end{aligned} \tag{2.33}$$

By our hypotheses on f, we have

$$f'(u(\lambda))u^2(\lambda) \geq \mu f(u(\lambda))u(\lambda) \quad \text{in } \Omega(\lambda). \tag{2.34}$$

Therefore,

$$(\mu - 1) \int_{\Omega(\lambda)} f(u(\lambda))u(\lambda)dx \leq C, \tag{2.35}$$

where the constant C depends only on λ. This constant can be chosen sufficiently large so that

$$\int_{\Omega \setminus \Omega(\lambda)} f(u(\lambda))u(\lambda)dx \leq C. \tag{2.36}$$

By (2.35), (2.36), and $\mu > 1$ it follows that there exists $C > 0$ independent of λ such that

$$\int_\Omega f(u(\lambda))u(\lambda)dx \leq C. \tag{2.37}$$

From here and from

$$\int_\Omega |\nabla u(\lambda)|^2 dx = \lambda \int_\Omega f(u(\lambda))u(\lambda)dx, \tag{2.38}$$

it follows that $u(\lambda)$ is bounded in $H_0^1(\Omega)$, independently with respect to λ. Consequently, up to a subsequence, we may suppose that there exists $u^* \in H_0^1(\Omega)$ such that

$$u(\lambda) \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega) \text{ if } \lambda \longrightarrow \lambda^*, \tag{2.39}$$

$$u(\lambda) \longrightarrow u^* \quad \text{a.e. in } \Omega \text{ if } \lambda \longrightarrow \lambda^*. \tag{2.40}$$

Hence

$$f(u(\lambda)) \longrightarrow f(u^*) \quad \text{a.e. in } \Omega \text{ if } \lambda \longrightarrow \lambda^*. \tag{2.41}$$

By (2.37) we get

$$\int_\Omega f(u(\lambda))dx \le C. \tag{2.42}$$

Since the mapping $\lambda \mapsto f(u(\lambda))$ is increasing and the integral is bounded, we find, by the monotone convergence theorem, $f(u^*) \in L^1(\Omega)$ and

$$f(u(\lambda)) \longrightarrow f(u^*) \quad \text{in } L^1(\Omega) \text{ if } \lambda \longrightarrow \lambda^*. \tag{2.43}$$

Let us now choose $v \in H_0^1(\Omega)$, $v \ge 0$. So,

$$\int_\Omega \nabla u(\lambda) \cdot \nabla v \, dx = \lambda \int_\Omega f(u(\lambda)) v \, dx. \tag{2.44}$$

On the other hand, we have already remarked that

$$\begin{gathered} f(u(\lambda))v \longrightarrow f(u^*)v \quad \text{a.e. in } \Omega, \text{ if } \lambda \longrightarrow \lambda^*, \\ \lambda \longmapsto f(u(\lambda))v \quad \text{is increasing}. \end{gathered} \tag{2.45}$$

By (2.39), it follows that

$$\int_\Omega f(u(\lambda))v \le C. \tag{2.46}$$

Now, again by the monotone convergence theorem,

$$\begin{gathered} f(u(\lambda))v \longrightarrow f(u^*)v \quad \text{in } L^1(\Omega) \text{ if } \lambda \longrightarrow \lambda^*, \\ \int_\Omega \nabla u^* \cdot \nabla v \, dx = \lambda^* \int_\Omega f(u^*)v \, dx. \end{gathered} \tag{2.47}$$

If $v \in H_0^1(\Omega)$ is arbitrary, we find the same conclusion if we consider $v = v^+ - v^-$. So, for every $v \in H_0^1(\Omega)$,

$$\begin{gathered} f(u(\lambda))v \longrightarrow f(u^*)v \quad \text{in } L^1(\Omega), \\ \int_\Omega \nabla u^* \cdot \nabla v \, dx = \lambda^* \int_\Omega f(u^*)v \, dx \end{gathered} \tag{2.48}$$

and $u^* \in H_0^1(\Omega)$. Consequently, u^* is a weak solution of problem (2.12). Moreover, for every $v \in H_0^1(\Omega)$,

$$\lambda^* \int_\Omega f'(u(\lambda))v^2 dx \leq \int_\Omega |\nabla v|^2 dx. \tag{2.49}$$

Applying again the monotone convergence theorem, we find that

$$f'(u^*)v^2 \in L^1(\Omega), \qquad \lambda^* \int_\Omega f'(u^*)v^2 dx \leq \int_\Omega |\nabla v|^2 dx. \tag{2.50}$$

□

The facts that

$$f(u^*) \in L^1(\Omega), \qquad f(u^*)u^* \in L^1(\Omega), \tag{2.51}$$

$$f'(u^*)v^2 \text{ in } L^1(\Omega) \quad \text{for every } v \in H_0^1(\Omega) \tag{2.52}$$

imply a supplementary regularity of u^*. In many concrete situations one may show that there exists n_0 such that, if $N \leq n_0$, then $u^* \in L^\infty(\Omega)$ and $f(u^*) \in L^\infty(\Omega)$.

We will deduce in what follows some known results, for the special case $f(t) = e^t$.

Theorem 2.8. *Let $f(t) = e^t$. Then*

$$f(u(\lambda)) \longrightarrow f(u^*) \quad \text{in } L^p(\Omega), \text{ if } \lambda \longrightarrow \lambda^*, \tag{2.53}$$

for every $p \in [1,5)$.

Consequently,

$$u(\lambda) \longrightarrow u^* \quad \text{in } W^{2,p}(\Omega), \text{ if } \lambda \longrightarrow \lambda^*, \tag{2.54}$$

for every $p \in [1,5)$.

Moreover, if $N \leq 9$, then

$$u^* \in L^\infty(\Omega), \qquad f(u^*) \in L^\infty(\Omega). \tag{2.55}$$

Proof. As we have already remarked, for every $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u(\lambda) \cdot \nabla v \, dx = \lambda \int_\Omega e^{u(\lambda)} v \, dx, \tag{2.56}$$

$$\lambda \int_\Omega e^{u(\lambda)} v^2 dx \leq \int_\Omega |\nabla v|^2 dx. \tag{2.57}$$

In (2.57) we choose $v = e^{(p-1)u(\lambda)} - 1 \in H_0^1(\Omega)$, for $p > 1$ arbitrary. We find

$$(p-1)\int_\Omega e^{(p-1)u(\lambda)} |\nabla u(\lambda)|^2 dx = \lambda \int_\Omega e^{u(\lambda)} [e^{(p-1)u(\lambda)} - 1] dx. \tag{2.58}$$

In (2.57) we put

$$v = e^{(p-1)/2u(\lambda)} - 1 \in H_0^1(\Omega). \tag{2.59}$$

Hence

$$\lambda \int_\Omega e^{u(\lambda)}\left[e^{(p-1)/2}u(\lambda) - 1\right]^2 dx \le \frac{(p-1)^2}{4}\int_\Omega e^{(p-1)u(\lambda)}\left|\nabla u(\lambda)\right|^2 dx. \tag{2.60}$$

Taking into account the relation (2.58), our relation (2.60) becomes

$$\begin{aligned}\lambda \int_\Omega e^{pu(\lambda)}dx - 2\lambda\int_\Omega e^{(p+1)/2}u(\lambda)dx + \lambda\int_\Omega e^{u(\lambda)}dx \\ \le \frac{p-1}{4}\left[\lambda\int_\Omega e^{pu(\lambda)}dx - \lambda\int_\Omega e^{u(\lambda)}dx\right].\end{aligned} \tag{2.61}$$

We now recall a basic inequality which will be used several times in our arguments.

Hölder's inequality. Let p and p' be dual indices, that is, $1/p + 1/p' = 1$ with $1 < p < \infty$. Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. Then $fg \in L^1(\Omega)$ and

$$\left|\int_\Omega f(x)g(x)dx\right| \le \left(\int_\Omega |f(x)|^p dx\right)^{1/p} \cdot \left(\int_\Omega |g(x)|^{p'} dx\right)^{1/p'}. \tag{2.62}$$

The particular case $p = p' = 2$ is known as the *Cauchy-Schwarz inequality*.

By Hölder's inequality, our relation (2.61) yields

$$\begin{aligned}\lambda\left(1 - \frac{p-1}{4}\right)\int_\Omega e^{pu(\lambda)}dx + \lambda\left(1 + \frac{p-1}{4}\right)\int_\Omega e^{u(\lambda)}dx \\ \le 2\lambda\int_\Omega e^{(p+1)/2}u(\lambda)dx \le C\left(\int_\Omega e^{pu(\lambda)}dx\right)^2,\end{aligned} \tag{2.63}$$

where C is a constant which does not depend on λ.

So, if $1 - (p-1)/4 > 0$, that is, $p < 5$, then the mapping $e^{u(\lambda)}$ is bounded in $L^p(\Omega)$.

We have already proved that if $\lambda \to \lambda^*$, then

$$e^{u(\lambda)} \longrightarrow e^{u^*} \quad \text{a.e. in } \Omega. \tag{2.64}$$

Moreover,

$$e^{u(\lambda)} \le e^{u^*} \quad \text{in } \Omega. \tag{2.65}$$

We recall in what follows a basic result in measure theory which will be applied several times in this book. We refer to Brezis [30] for a complete proof and further comments.

Lebesgue's dominated convergence theorem. Let $f_n : \mathbb{R}^N \to \mathbb{R}$ be a sequence of functions in $L^1(\mathbb{R}^N)$. We assume that

(i) $f_n(x) \to f(x)$ a.e. in $\mathbb{R}^N$,
(ii) there exists $g \in L^1(\mathbb{R}^N)$ such that, for all $n \ge 1$, $|f_n(x)| \le g(x)$ a.e. in $\mathbb{R}^N$.

Then $f \in L^1(\mathbb{R}^N)$ and $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$.

By the Lebesgue dominated convergence theorem, we find that for every $1 \leq p < 5$,

$$\begin{aligned} e^{u^*} &\in L^p(\Omega), \\ e^{u(\lambda)} &\longrightarrow e^{u^*} \quad \text{in } L^p(\Omega). \end{aligned} \tag{2.66}$$

Taking into account the relation (2.12) which is fulfilled by $u(\lambda)$, and using a standard regularity theorem for elliptic equations, it follows that

$$u(\lambda) \longrightarrow u^* \text{ in } W^{2,p}(\Omega) \quad \text{if } \lambda \longrightarrow \lambda^*. \tag{2.67}$$

On the other hand, by Sobolev inclusions,

$$W^{2,p} \subset L^\infty(\Omega) \quad \text{if } N < 2p. \tag{2.68}$$

So, if $N \leq 9$,

$$u(\lambda) \longrightarrow u^* \text{ in } L^\infty(\Omega) \quad \text{if } \lambda \longrightarrow \lambda^*. \tag{2.69}$$

Hence

$$u^* \in L^\infty(\Omega), \qquad e^{u^*} \in L^\infty(\Omega). \tag{2.70}$$

This concludes our proof. □

Remark 2.9. The above result is optimal, since Joseph and Lundgren showed in [114] that if $N = 10$ and Ω is an open ball, then $u^* \notin L^\infty(\Omega)$.

2.5. Bifurcation for the logistic equation on the whole space

Nevertheless, each time I can, I aim the absolute rigor for two reasons. In the first place, it is always hard for a geometer to consider a problem without resolving it completely. In the second place, these equations that I will study are susceptible, not only to physical applications, but also to analytical applications. And who will say that the other problems of mathematical physics will not, one day, be called to play in analysis a considerable role, as has been the case of the most elementary of them?

Henri Poincaré

In this section, we are concerned with the existence, uniqueness, or the nonexistence of positive solutions of the eigenvalue logistic problem with absorption

$$-\Delta u = \lambda\big(V(x)u - f(u)\big) \quad \text{in } \mathbb{R}^N,\ N \geq 3, \tag{2.71}$$

where V is a smooth sign-changing potential and $f : [0,\infty) \rightarrow [0,\infty)$ is a smooth function. Equations of this type arise in the study of population dynamics. In this case, the unknown u corresponds to the density of a population, the potential V describes the birth rate of the population, while the term $-f(u)$ in (2.71) signifies the fact that the population is self-limiting. In the region where V is positive (resp., negative) the population has positive (resp., negative) birth rate. Since u describes a population density, we are interested in investigating only positive solutions of problem (2.71).

Our main results in this section, but also in other chapters of this book, concern some classes of nonlinear eigenvalue problems associated to linear or quasilinear elliptic operators. Our interest for *spectral problems* is also motivated by the following quotation of Gould [100] which asserts, in fact, that the mathematical spectrum is partly made of "eigenvalues," a strange word which has not been immediately adopted: *The concept of an eigenvalue is of great importance in both pure and applied mathematics. The German word "eigen" means "characteristic" and the hybrid word eigenvalue is used for characteristic numbers in order to avoid confusion with the many other uses in English of the word "characteristic." There can be no doubt that "eigenvalue" will soon find its way into the standard dictionaries. The English language has many such hybrids: for example "liverwurst."* We conclude these historical comments with the following deep remarks which are due to M. Zworski [225]: *Eigenvalues describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life* $[\cdots]$. *In most situation however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate. Eigenvalues are yet another expression of humanity's narcissist desire for immortality.*

Our results are related to a certain linear eigenvalue problem. We recall in what follows the results that we need in the sequel. Let Ω be an arbitrary open set in $\mathbb{R}^N$, $N \geq 3$. Consider the eigenvalue problem

$$-\Delta u = \lambda V(x)u \quad \text{in } \Omega,\ u \in H_0^1(\Omega). \tag{2.72}$$

Problems of this type have a long history. If Ω is bounded and $V \equiv 1$, problem (2.72) is related to the Riesz-Fredholm theory of self adjoint and compact operators (see, e.g., [30, Theorem VI.11] by Brezis). The case of a nonconstant potential V has been first considered in the pioneering papers of Bocher [27], Hess and Kato [104], Minakshisundaran and Pleijel [176], and Pleijel [150]. For instance, Minakshisundaran and Pleijel [150, 176] studied the case where Ω is bounded, $V \in L^\infty(\Omega)$, $V \geq 0$ in Ω, and $V > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of (2.72) if Ω is not necessarily bounded has been given by Szulkin and Willem [217] under the assumption that the sign-changing potentialsign-changing potential V satisfies

$$\begin{gathered} V \in L^1_{\text{loc}}(\Omega), \qquad V^+ = V_1 + V_2 \not\equiv 0, \qquad V_1 \in L^{N/2}(\Omega), \\ \lim_{\substack{x \to y \\ x \in \Omega}} |x-y|^2 V_2(x) = 0 \quad \text{for every } y \in \overline{\Omega}, \qquad \lim_{\substack{|x| \to \infty \\ x \in \Omega}} |x|^2 V_2(x) = 0. \end{gathered} \tag{H}$$

We have denoted $V^+(x) = \max\{V(x), 0\}$. Obviously, $V = V^+ - V^-$, where $V^-(x) = \max\{-V(x), 0\}$.

In order to find the principal eigenvalue of (2.72), Szulkin and Willem [217] proved that the minimization problem

$$\min\left\{\int_\Omega |\nabla u|^2 dx;\ u \in H_0^1(\Omega),\ \int_\Omega V(x)u^2 dx = 1\right\} \tag{2.73}$$

has a solution $\varphi_1 = \varphi_1(\Omega) \geq 0$ which is an eigenfunction of (2.72) corresponding to the eigenvalue $\lambda_1(\Omega) = \int_\Omega |\nabla \varphi_1|^2 dx$.

Throughout this section, the sign-changing potential sign-changing potential $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to be a Hölder function that satisfies

$$\begin{aligned} V \in L^\infty(\mathbb{R}^N), \qquad & V^+ = V_1 + V_2 \neq 0, \\ V_1 \in L^{N/2}(\mathbb{R}^N), \qquad & \lim_{|x|\to\infty} |x|^2 V_2(x) = 0. \end{aligned} \tag{V}$$

We suppose that the nonlinear absorption term $f : [0,\infty) \to [0,\infty)$ is a C^1-function such that

(f1) $f(0) = f'(0) = 0$ and $\liminf_{u\searrow 0} f'(u)/u > 0$;
(f2) the mapping $f(u)/u$ is increasing in $(0,+\infty)$.

This assumption implies $\lim_{u\to+\infty} f(u) = +\infty$. We impose that f does not have a sublinear growth at infinity. More precisely, we assume

(f3) $\lim_{u\to+\infty} f(u)/u > \|V\|_{L^\infty}$.

Our framework includes the following cases: (i) $f(u) = u^2$ that corresponds to the Fisher equation (see Fisher [80]) and the Kolmogoroff-Petrovsky-Piscounoff equation [123] (see also Kazdan and Warner [117] for a comprehensive treatment of these equations); (ii) $f(u) = u^{(N+2)/(N-2)}$ (for $N \geq 6$) which is related to the conform scalar curvature equation, cf. Li and Ni [135].

2.5.1. Exact value of the bifurcation parameter

For any $R > 0$, denote $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$ and set

$$\lambda_1(R) = \min\left\{ \int_{B_R} |\nabla u|^2 dx;\ u \in H_0^1(B_R),\ \int_{B_R} V(x)u^2 dx = 1 \right\}. \tag{2.74}$$

Consequently, the mapping $R \mapsto \lambda_1(R)$ is decreasing and so, there exists

$$\Lambda := \lim_{R\to\infty} \lambda_1(R) \geq 0. \tag{2.75}$$

We first state a sufficient condition so that Λ is positive. For this aim we impose the additional assumptions

$$\text{there exist } A, \alpha > 0 \text{ such that } V^+(x) \leq A|x|^{-2-\alpha} \quad \forall x \in \mathbb{R}^N, \tag{2.76}$$

$$\lim_{x\to 0} |x|^{2(N-1)/N} V_2(x) = 0. \tag{2.77}$$

Theorem 2.10. *Assume that V satisfies conditions (V), (2.76), and (2.77). Then $\Lambda > 0$.*

Proof. For any $R > 0$, we fix $u \in H_0^1(B_R)$ such that $\int_{B_R} V(x)u^2 dx = 1$. We have

$$1 = \int_{B_R} V(x)u^2 dx \leq \int_{B_R} V^+(x)u^2 dx = \int_{B_R} V_1(x)u^2 dx + \int_{B_R} V_2(x)u^2 dx. \tag{2.78}$$

Since $V_1 \in L^{N/2}(\mathbb{R}^N)$, using the Cauchy-Schwarz inequality and Sobolev embeddings we obtain

$$\int_{B_R} V_1(x)u^2 dx \le ||V_1||_{L^{N/2}(B_R)}\|u\|^2_{L^{2^*}(B_R)} \le C_1||V_1||_{L^{N/2}(\mathbb{R}^N)} \int_{B_R} |\nabla u|^2 dx, \tag{2.79}$$

where 2^* denotes the critical Sobolev exponent, that is, $2^* = 2N/(N-2)$.

Fix $\epsilon > 0$. By our assumption (V), there exist positive numbers δ, R_1, and R such that $R^{-1} < \delta < R_1 < R$ such that for all $x \in B_R$ satisfying $|x| \ge R_1$ we have

$$|x|^2 V_2(x) \le \epsilon. \tag{2.80}$$

On the other hand, by (V), for any $x \in B_R$ with $|x| \le \delta$ we have

$$|x|^{2(N-1)/N} V_2(x) \le \epsilon. \tag{2.81}$$

Define $\Omega := \omega_1 \cup \omega_2$, where $\omega_1 := B_R \setminus \overline{B}_{R_1}$, $\omega_2 := B_\delta \setminus \overline{B}_{1/R}$, and $\omega := B_{R_1} \setminus \overline{B}_\delta$.

We need in what follows the following celebrated inequality which is due to Hardy (see [101]) and which can be viewed as an extension of the Poincaré inequality in the case of singular potentials.

Hardy's inequality. Assume that $1 < p < N$. Then

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx \le \frac{p^p}{(N-p)^p} \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \tag{2.82}$$

for any $u \in W^{1,p}(\mathbb{R}^N)$ such that $u/|x| \in L^p(\mathbb{R}^N)$. Moreover, the constant $p^p(N-p)^{-p}$ is optimal.

By (2.80) and Hardy's inequality we find

$$\int_{\omega_1} V_2(x)u^2 dx \le \epsilon \int_{\omega_1} \frac{u^2}{|x|^2} dx \le C_2\epsilon \int_{B_R} |\nabla u|^2 dx. \tag{2.83}$$

Using now (2.81) and Hölder's inequality we obtain

$$\begin{aligned}
\int_{\omega_2} V_2(x)u^2 dx &\le \epsilon \int_{\omega_2} \frac{u^2}{|x|^{2(N-1)/N}} dx \\
&\le \epsilon \left[\int_{\omega_2} \left(\frac{1}{|x|^{2(N-1)/N}} dx \right)^{N/2} dx \right]^{2/N} \|u\|^2_{L^{2^\star}(B_R)} \\
&\le C\epsilon \left(\int_{1/R}^{\delta} \frac{1}{s^{N-1}} s^{N-1} \omega_N ds \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\
&\le C_3 \left(\delta - \frac{1}{R} \right)^{2/N} \int_{B_R} |\nabla u|^2 dx.
\end{aligned} \tag{2.84}$$

By compactness and our assumption (V), there exists a finite covering of $\overline{\omega}$ by the closed balls $\overline{B}_{r_1}(x_1), \dots, \overline{B}_{r_k}(x_k)$ such that for all $1 \le j \le k$,

$$\text{if } |x - x_j| \le r_j \text{ then } |x - x_j|^{2(N-1)/N} V_2(x) \le \epsilon. \tag{2.85}$$

There exists $r > 0$ such that for any $1 \leq j \leq k$,

$$\text{if } |x - x_j| \leq r \text{ then } |x - x_j|^{2(N-1)/N} V_2(x) \leq \frac{\epsilon}{k}. \tag{2.86}$$

Define $A := \cup_{j=1}^{k} B_r(x_j)$. The above estimate, Hölder's inequality, and Sobolev embeddings yield

$$\begin{aligned}
\int_{B_r(x_j)} V_2(x) u^2 dx &\leq \frac{\epsilon}{k} \int_{B_r(x_j)} \frac{u^2}{|x - x_j|^{2(N-1)/N}} dx \\
&\leq \frac{\epsilon}{k} \left[\int_{B_r(x_j)} \left(|x - x_j|^{-2(N-1)/N} \right)^{N/2} dx \right]^{2/N} \|u\|^2_{L^{2^\star}(B_R)} \\
&\leq C \frac{\epsilon}{k} \left(\int_{B_r} \frac{1}{|x|^{N-1}} dx \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\
&= C \frac{\epsilon}{k} \left(\int_0^r \frac{1}{s^{N-1}} s^{N-1} \omega_N ds \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\
&= C' \int_{B_R} |\nabla u|^2 dx,
\end{aligned} \tag{2.87}$$

for any $j = 1, \ldots, k$. By addition we find

$$\int_A V_2(x) u^2 dx \leq C_4 \int_{B_R} |\nabla u|^2 dx. \tag{2.88}$$

It follows from (2.85) that $V_2 \in L^\infty(\omega \setminus A)$. Actually, if $x \in \omega \setminus A$ it follows that there exists $j \in \{1, \ldots, k\}$ such that $r_j > |x - x_j| > r > 0$. Thus

$$V_2(x) \leq r^{-2(N-1)/N} \epsilon. \tag{2.89}$$

Hence

$$\int_{\omega \setminus A} V_2(x) u^2 dx \leq \epsilon r^{-2(N-1)/N} \int_{\omega \setminus A} u^2 dx \leq C_5 \int_{B_R} |\nabla u|^2 dx. \tag{2.90}$$

Now, by inequalities (2.79), (2.83), (2.84), (2.88), and (2.90) we have

$$\lambda_1(R) \geq \left\{ C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2 \epsilon + C_3 \left(\delta - R^{-1}\right)^{2/N} + C_4 + C_5 \right\}^{-1}, \tag{2.91}$$

and passing to the limit as $R \to \infty$ we conclude that

$$\Lambda \geq \left(C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2 \epsilon + C_3 \delta^{2/N} + C_4 + C_5 \right)^{-1} > 0. \tag{2.92}$$

This completes the proof of Theorem 2.10. □

2.5.2. Bifurcation of entire solutions

Our main result asserts that the real number $\Lambda > 0$ defined by

$$\Lambda := \lim_{R \to \infty} \lambda_1(R) \tag{2.93}$$

plays a crucial role for the nonlinear eigenvalue logistic problem

$$
\begin{gathered}
-\Delta u=\lambda(V(x)u-f(u)) \quad \text{in } \mathbb{R}^N,\\
u>0 \quad \text{in } \mathbb{R}^N,\\
\lim_{|x|\to\infty} u(x)=0.
\end{gathered} \tag{2.94}
$$

The following existence and nonexistence results show that Λ serves as a bifurcation point in our problem (2.94).

Theorem 2.11. *Assume that V and f satisfy the assumptions* (V), (2.76), $(f1)$, $(f2)$, *and* $(f3)$.

Then the following hold:

(i) *problem* (2.94) *has a unique solution for any $\lambda > \Lambda$;*
(ii) *problem* (2.94) *does not have any solution for all $\lambda \le \Lambda$.*

The additional condition (2.76) implies that $V^+ \in L^{N/2}(\mathbb{R}^N)$, which does not follow from the basic hypothesis (V). As we will see in the next section, this growth assumption is essential in order to establish the existence of positive solutions of (2.71) *decaying to zero* at infinity.

In particular, Theorem 2.11 shows that if $V(x) < 0$ for sufficiently large $|x|$ (i.e., if the population has negative birth rate) then any positive solution (i.e., the population density) of (2.71) tends to zero as $|x| \to \infty$.

Before giving the proof of Theorem 2.11, we show in what follows that the logistic equation (2.71) has entire positive solutions if λ is sufficiently large. However, we are not able to establish that this solution decays to zero at infinity. This will be proved thereafter by means of the additional assumption (2.76). More precisely, we have the following auxiliary result.

Proposition 2.12. *Assume that the functions V and f satisfy conditions* (V), $(f1)$, $(f2)$, *and* $(f3)$. *Then the problem*

$$
\begin{gathered}
-\Delta u=\lambda(V(x)u-f(u)) \quad \text{in } \mathbb{R}^N,\\
u>0 \quad \text{in } \mathbb{R}^N
\end{gathered} \tag{2.95}
$$

has at least one solution, for any $\lambda > \Lambda$.

Proof. For any $R > 0$, consider the boundary value problem

$$
\begin{gathered}
-\Delta u=\lambda(V(x)u-f(u)) \quad \text{in } B_R,\\
u>0 \quad \text{in } B_R,\\
u=0 \quad \text{on } \partial B_R.
\end{gathered} \tag{2.96}
$$

We first prove that problem (2.96) has at least one solution, for any $\lambda > \lambda_1(R)$. Indeed, the function $\overline{u}(x) = M$ is a supersolution of (2.96), for any M large enough. This follows from $(f3)$ and the boundedness of V. Next, in order to find a positive subsolution, we consider the minimization problem

$$\min_{u\in H_0^1(B_R)} \int_{B_R} \left(|\nabla u|^2 - \lambda V(x)u^2\right)dx. \tag{2.97}$$

Since $\lambda > \lambda_1(R)$, it follows that the least eigenvalue μ_1 is negative. Moreover, the corresponding eigenfunction e_1 satisfies

$$\begin{aligned} -\Delta e_1 - \lambda V(x)e_1 &= \mu_1 e_1 \quad \text{in } B_R, \\ e_1 &> 0 \quad \text{in } B_R, \\ e_1 &= 0 \quad \text{on } \partial B_R. \end{aligned} \tag{2.98}$$

Then the function $\underline{u}(x) = \varepsilon e_1(x)$ is a subsolution of problem (2.96). Indeed, it is enough to check that

$$-\Delta(\varepsilon e_1) - \lambda\varepsilon V e_1 + \lambda f(\varepsilon e_1) \le 0 \quad \text{in } B_R, \tag{2.99}$$

that is, by (2.98),

$$\varepsilon\mu_1 e_1 + \lambda f(\varepsilon e_1) \le 0 \quad \text{in } B_R. \tag{2.100}$$

But

$$f(\varepsilon e_1) = \varepsilon f'(0)e_1 + \varepsilon e_1 o(1), \quad \text{as } \varepsilon \to 0. \tag{2.101}$$

So, since $f'(0) = 0$, relation (2.100) becomes

$$\varepsilon e_1(\mu_1 + o(1)) \le 0 \tag{2.102}$$

which is true, provided $\varepsilon > 0$ is small enough, due to the fact that $\mu_1 < 0$.

Fix $\lambda > \Lambda$ and an arbitrary sequence $R_1 < R_2 < \cdots < R_n < \cdots$ of positive numbers such that $R_n \to \infty$ and $\lambda_1(R_1) < \lambda$. Let u_n be the solution of (2.96) on B_{R_n}. Fix a positive number M such that $f(M)/M > \|V\|_{L^\infty(\mathbb{R}^N)}$. The above arguments show that we can assume $u_n \le M$ in B_{R_n}, for any $n \ge 1$. Since u_{n+1} is a supersolution of (2.96) for $R = R_n$, we can also assume that $u_n \le u_{n+1}$ in B_{R_n}. Thus the function $u(x) := \lim_{n\to\infty} u_n(x)$ exists and is well-defined and positive in $\mathbb{R}^N$. Standard elliptic regularity arguments imply that u is a solution of problem (2.95). □

The above result shows the importance of the assumption (2.76) in the statement of Theorem 2.11. Indeed, assuming that V satisfies only hypothesis (V), it is not clear whether or not the solution constructed in the proof of Proposition 2.12 tends to 0 as $|x| \to \infty$. However, it is easy to observe that if $\lambda > \Lambda$ and V satisfies (2.76) then problem (2.94) has at least one solution. Indeed, we first observe that

$$\underline{u}(x) = \begin{cases} \varepsilon e_1(x) & \text{if } x \in B_R, \\ 0 & \text{if } x \notin B_R \end{cases} \tag{2.103}$$

is a subsolution of problem (2.94), for some fixed $R > 0$, where e_1 satisfies (2.98). Next, we observe that $\overline{u}(x) = n/(1 + |x|^2)$ is a supersolution of (2.94). Indeed, $\overline{u}$ satisfies

$$-\Delta\overline{u}(x) = \frac{2\left[n(1 + |x|^2) - 4|x|^2\right]}{\left(1 + |x|^2\right]^2} u(x), \quad x \in \mathbb{R}^N. \tag{2.104}$$

It follows that $\overline{u}$ is a supersolution of (2.94) provided

$$\frac{2\left[n(1 + |x|^2) - 4|x|^2\right]}{\left(1 + |x|^2\right)^2} \geq \lambda V(x) - \lambda f\left(\frac{n}{1 + |x|^2}\right), \quad x \in \mathbb{R}^N. \tag{2.105}$$

This inequality follows from $(f3)$ and (2.76), provided that n is large enough.

We split the proof of Theorem 2.11 into several steps.

Proposition 2.13. *Let u be an arbitrary solution of problem* (2.94)*. Then there exists $C > 0$ such that $|u(x)| \leq C|x|^{2-N}$ for all $x \in \mathbb{R}^N$.*

Proof. Let ω_N be the surface area of the unit sphere in $\mathbb{R}^N$. Consider the function V^+u as a Newtonian potential and define

$$v(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{V^+(y)u(y)}{|x - y|^{N-2}} dy. \tag{2.106}$$

A straightforward computation shows that

$$-\Delta v = V^+(x)u \quad \text{in } \mathbb{R}^N. \tag{2.107}$$

But, by (2.76) and since u is bounded,

$$V^+(y)u(y) \leq C|y|^{-2-\alpha} \quad \forall y \in \mathbb{R}^N. \tag{2.108}$$

So, by Lemma 2.3 in Li and Ni [135],

$$v(x) \leq C|x|^{-\alpha}, \quad \forall x \in \mathbb{R}^N, \tag{2.109}$$

provided that $\alpha < N - 2$. Set $w(x) = Cv(x) - u(x)$. Hence $w(x) \to 0$ as $|x| \to \infty$. Let us choose C sufficiently large so that $w(0) > 0$. We claim that this implies

$$w(x) > 0 \quad \forall x \in \mathbb{R}^N. \tag{2.110}$$

Indeed, if not, let $x_0 \in \mathbb{R}^N$ be a local minimum point of w. This means that $w(x_0) < 0$, $\nabla w(x_0) = 0$ and $\Delta w(x_0) \geq 0$. But

$$\Delta w(x_0) = -CV^+(x_0)u(x_0) + \lambda\left(V(x_0)u(x_0) - f(u(x_0))\right) < 0, \tag{2.111}$$

provided that $C > \lambda$. This contradiction implies (2.110). Consequently,

$$u(x) \leq Cv(x) \leq C|x|^{-\alpha}, \quad \text{for any } x \in \mathbb{R}^N. \tag{2.112}$$

So, using again (2.76),

$$V^+(x)u(x) \le C|x|^{-2-2\alpha} \quad \forall x \in \mathbb{R}^N. \tag{2.113}$$

Lemma 2.3 in [135] by Li and Ni yields the improved estimate

$$v(x) \le C|x|^{-2\alpha} \quad \forall x \in \mathbb{R}^N, \tag{2.114}$$

provided that $2\alpha < N-2$, and so on. Let n_α be the largest integer such that $n_\alpha \alpha < N-2$. Repeating $n_\alpha + 1$ times the above argument based on Lemma 2.3(i) and (iii) in [135] by Li and Ni, we obtain

$$u(x) \le C|x|^{2-N} \quad \forall x \in \mathbb{R}^N. \tag{2.115}$$

This completes the proof of Proposition 2.13. □

Proposition 2.14. *Let u be a solution of problem* (2.94)*. Then V^+u, V^-u, $f(u) \in L^1(\mathbb{R}^N)$, and $u \in H^1(\mathbb{R}^N)$.*

Proof. For any $R > 0$, consider the average function

$$\overline{u}(R) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B_R} u(x)d\sigma = \frac{1}{\omega_N} \int_{\partial B_1} u(rx)d\sigma, \tag{2.116}$$

where ω_N denotes the surface area of S^{N-1}. Then

$$\overline{u}'(R) = \frac{1}{\omega_N} \int_{\partial B_1} \frac{\partial u}{\partial \nu}(rx)d\sigma = \frac{1}{\omega_N R^{N-1}} \int_{\partial B_R} \frac{\partial u}{\partial \nu}(x)d\sigma = \frac{1}{\omega_N R^{N-1}} \int_{B_R} \Delta u(x)dx. \tag{2.117}$$

Hence

$$\begin{aligned} \omega_N R^{N-1}\overline{u}'(R) &= -\lambda \int_{B_R} (V(x)u - f(u))\,dx \\ &= -\lambda \int_{B_R} V^+(x)u\,dx + \lambda \int_{B_R} (V^-(x)u + f(u))\,dx. \end{aligned} \tag{2.118}$$

By Proposition 2.13, there exists $C > 0$ such that $|\overline{u}(r)| \le Cr^{-N+2}$, for any $r > 0$. So, by (2.76),

$$\int_{1 \le |x| \le r} V^+(x)u\,dx \le CA \int_{1 \le |x| \le r} |x|^{-N-\alpha}dx \le C, \tag{2.119}$$

where C does not depend on r. This implies $V^+u \in L^1(\mathbb{R}^N)$.

By contradiction, assume that $V^-u + f(u) \notin L^1(\mathbb{R}^N)$. So, by (2.118), $\overline{u}'(r) > 0$ if r is sufficiently large. It follows that $\overline{u}(r)$ does not converge to 0 as $r \to \infty$, which contradicts Proposition 2.13. So, $V^-u + f(u) \in L^1(\mathbb{R}^N)$. Next, in order to establish that $u \in L^2(\mathbb{R}^N)$, we observe that our assumption $(f1)$ implies the existence of some positive numbers a and δ such that $f'(t) > at$, for any $0 < t < \delta$. This implies $f(t) > at^2/2$, for any $0 < t < \delta$. Since u decays to 0 at infinity, it follows that the set $\{x \in \mathbb{R}^N;\ u(x) \ge \delta\}$ is compact.

Hence

$$\int_{\mathbb{R}^N} u^2 dx = \int_{[u \geq \delta]} u^2 dx + \int_{[u<\delta]} u^2 dx \leq \int_{[u \geq \delta]} u^2 dx + \frac{2}{a} \int_{[u<\delta]} f(u) dx < +\infty, \quad (2.120)$$

since $f(u) \in L^1(\mathbb{R}^N)$.

It remains to prove that $\nabla u \in L^2(\mathbb{R}^N)^N$. We first observe that after multiplication by u in (2.71) and integration we find

$$\int_{B_R} |\nabla u|^2 dx - \int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma = \lambda \int_{B_R} (V(x)u - f(u)) dx, \quad (2.121)$$

for any $r > 0$. Since $Vu - f(u) \in L^1(\mathbb{R}^N)$, it follows that the left-hand side has a finite limit as $r \to \infty$. Arguing by contradiction and assuming that $\nabla u \notin L^2(\mathbb{R}^N)^N$, it follows that there exists $R_0 > 0$ such that

$$\int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma \geq \frac{1}{2} \int_{B_R} |\nabla u|^2 dx, \quad \text{for any } R \geq R_0. \quad (2.122)$$

Define the functions

$$A(R) = \int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma, \qquad B(R) = \int_{\partial B_R} u^2(x) d\sigma,$$
$$C(R) = \int_{B_R} \left| \nabla u(x) \right|^2 dx. \quad (2.123)$$

Relation (2.122) can be rewritten as

$$A(R) \geq \frac{1}{2} C(R), \quad \text{for any } R \geq R_0. \quad (2.124)$$

On the other hand, by the Cauchy-Schwarz inequality,

$$A^2(R) \leq \left(\int_{\partial B_R} u^2 d\sigma \right) \left(\int_{\partial B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma \right) \leq B(R) C'(R). \quad (2.125)$$

Using now (2.124) we obtain

$$C'(R) \geq \frac{C^2(R)}{4B(R)}, \quad \text{for any } R \geq R_0. \quad (2.126)$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{4}{C(r)} + \int_0^r \frac{dt}{B(t)} \right]_{r=R} \leq 0, \quad \text{for any } R \geq R_0. \quad (2.127)$$

But, since $u \in L^2(\mathbb{R}^N)$, it follows that $\int_0^\infty B(t) dt$ converges, so

$$\lim_{R \to \infty} \int_0^R \frac{dt}{B(t)} = +\infty. \quad (2.128)$$

On the other hand, our assumption $|\nabla u| \notin L^2(\mathbb{R}^N)$ implies

$$\lim_{R\to\infty} \frac{1}{C(R)} = 0. \tag{2.129}$$

Relations (2.127), (2.128), and (2.129) yield a contradiction, so our proof is complete. □

Proposition 2.15. *Let u and v be two distinct solutions of problem* (2.94). *Then*

$$\lim_{R\to\infty} \int_{\partial B_R} u(x)\frac{\partial v}{\partial \nu}(x)d\sigma = 0. \tag{2.130}$$

Proof. By multiplication with v in (2.94) and integration on B_R, we find

$$\int_{B_R} \nabla u \cdot \nabla v\, dx - \int_{\partial B_R} u\frac{\partial v}{\partial \nu} d\sigma = \lambda \int_{B_R} (V(x)uv - f(u)v)\,dx. \tag{2.131}$$

So, by Proposition 2.14, there exists and is finite $\lim_{R\to\infty} \int_{\partial B_R} u(\partial v/\partial \nu)d\sigma$. But, by the Cauchy-Schwarz inequality,

$$\left| \int_{\partial B_R} u\frac{\partial v}{\partial \nu} d\sigma \right| \le \left(\int_{\partial B_R} u^2 d\sigma \right)^{1/2} \left(\int_{\partial B_R} |\nabla v|^2 d\sigma \right)^{1/2}. \tag{2.132}$$

Since $u, |\nabla v| \in L^2(\mathbb{R}^N)$, it follows that $\int_0^\infty (\int_{\partial B_R} (u^2 + |\nabla v|^2) d\sigma) dx$ is convergent. Hence

$$\lim_{R\to\infty} \int_{\partial B_R} (u^2 + |\nabla v|^2)\, d\sigma = 0. \tag{2.133}$$

Our conclusion now follows by (2.132) and (2.133). □

Proof of Theorem 2.11. (i) The existence of a solution follows with the arguments given in the preceding section. In order to establish the uniqueness, let u and v be two solutions of (2.94). We can assume without loss of generality that $u \le v$. This follows from the fact that $\overline{u} = \min\{u, v\}$ is a supersolution of (2.94) and $\underline{u}$ defined in (2.103) is an arbitrary small subsolution. So, it sufficient to consider the ordered pair consisting of the corresponding solution and v.

Since u and v are solutions we have, by Green's formula,

$$\int_{\partial B_R} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma = \lambda \int_{B_R} uv\left(\frac{f(v)}{v} - \frac{f(u)}{u} \right) dx. \tag{2.134}$$

By Proposition 2.15, the left-hand side converges to 0 as $R \to \infty$. So, (f1) and our assumption $u \le v$ force $u = v$ in $\mathbb{R}^N$.

(ii) By contradiction, let $\lambda \le \Lambda$ be such that problem (2.94) has a solution for this λ. So

$$\int_{B_R} |\nabla u|^2 dx - \int_{\partial B_R} u\frac{\partial u}{\partial \nu} d\sigma = \lambda \int_{B_R} (V(x)u^2 - f(u)u)\,dx. \tag{2.135}$$

By Propositions 2.14 and 2.15 and letting $R \to \infty$, we find

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx < \lambda \int_{\mathbb{R}^N} V(x)u^2 dx. \tag{2.136}$$

On the other hand, using the definition of Λ and (2.74), we obtain

$$\Lambda \int_{\mathbb{R}^N} V\zeta^2 dx \leq \int_{\mathbb{R}^N} |\nabla \zeta|^2 dx, \tag{2.137}$$

for any $\zeta \in C_0^2(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} V\zeta^2 dx > 0$.

Fix $\zeta \in C_0^2(\mathbb{R}^N)$ such that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if $|x| \leq 1$, and $\zeta(x) = 0$ if $|x| \geq 2$. For any $n \geq 1$ define $\Psi_n(x) = \zeta_n(x)u(x)$, where $\zeta_n(x) = \zeta(|x|/n)$. Thus $\Psi_n(x) \to u(x)$ as $n \to \infty$, for any $x \in \mathbb{R}^N$. Since $u \in H^1(\mathbb{R}^N)$, it follows by [30, Corollary IX.13] by Brezis that $u \in L^{2N/(N-2)}(\mathbb{R}^N)$. So, the Lebesgue dominated convergence theorem yields

$$\Psi_n \longrightarrow u \quad \text{in } L^{2N/(N-2)}(\mathbb{R}^N). \tag{2.138}$$

We claim that

$$\nabla\Psi_n \longrightarrow \nabla u \quad \text{in } L^2(\mathbb{R}^N)^N. \tag{2.139}$$

Indeed, let $\Omega_n := \{x \in \mathbb{R}^N;\ n < |x| < 2n\}$. Applying Hölder's inequality we find

$$\begin{aligned} \|\nabla\Psi_n - \nabla u\|_{L^2(\mathbb{R}^N)} &\leq \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\nabla\zeta_n\|_{L^2(\Omega_n)} \\ &\leq \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^{2N/(N-2)}(\Omega_n)} \cdot \|\nabla\zeta_n\|_{L^N(\mathbb{R}^N)}. \end{aligned} \tag{2.140}$$

But, since $|\nabla u| \in L^2(\mathbb{R}^N)$, it follows by Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} = 0. \tag{2.141}$$

Next, we observe that

$$\|\nabla\zeta_n\|_{L^N(\mathbb{R}^N)} = \|\nabla\zeta\|_{L^N(\mathbb{R}^N)}. \tag{2.142}$$

Since $u \in L^{2N/(N-2)}(\mathbb{R}^N)$ then

$$\lim_{n\to\infty} \|u\|_{L^{2N/(N-2)}(\Omega_n)} = 0. \tag{2.143}$$

Relations (2.140)–(2.143) imply our claim (2.139).

Since $V^\pm u^2 \in L^1(\mathbb{R}^N)$ and $V^\pm\Psi_n^2 \leq V^\pm u^2$, it follows by Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V^\pm\Psi_n^2 dx = \int_{\mathbb{R}^N} V^\pm u^2 dx. \tag{2.144}$$

Consequently,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V\Psi_n^2 dx = \int_{\mathbb{R}^N} Vu^2 dx. \tag{2.145}$$

So, by (2.136) and (2.145), it follows that there exists $n_0 \geq 1$ such that

$$\int_{\mathbb{R}^N} V\Psi_n^2 dx > 0, \quad \text{for any } n \geq n_0. \tag{2.146}$$

This means that we can write (2.137) for ζ replaced by $\Psi_n \in C_0^2(\mathbb{R}^N)$. Using then (2.139) and (2.145) we find

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \Lambda \int_{\mathbb{R}^N} V u^2 dx. \tag{2.147}$$

Relations (2.136) and (2.147) yield a contradiction, so problem (2.94) has no solution if $\lambda \leq \Lambda$. □

2.6. Bibliographical notes

Augustin-Louis Cauchy is generally acknowledged as the discoverer of the implicit function theorem. However, the germs of this theorem appeared even in the works of Sir Isaac Newton (1642–1727) and Gotffried Leibniz. Major advances of the implicit function theorem are recently due to John Nash [165] and Jürgen Moser [157], in connection with a big open problem of the last century: *can an arbitrary Riemannian manifold be isometrically imbedded in Euclidean space?*

The implicit function theorem is applied in this section to prove Theorem 15, which is a basic property in bifurcation theory. The roots of this theorem go back to the papers by Keller and Cohen (see, e.g., [119]). Amann [9] applied this abstract result in many concrete problems involving nonlinear partial differential equations of elliptic type.

3 Nonsmooth mountain pass theory

The strongest explosive is neither toluene nor the atomic bomb, but the human idea.
Grigore Moisil (1906–1973)

3.1. Basic properties of locally Lipschitz functionals

Throughout this chapter, X denotes a real Banach space. Let X^* be its dual and, for every $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle$ be the duality pairing between X^* and X.

Definition 3.1. A functional $f : X \to \mathbb{R}$ is said to be locally Lipschitz provided that, for every $x \in X$, there exists a neighborhood V of x and a positive constant $k = k(V)$ depending on V such that

$$|f(y) - f(z)| \leq k\|y - z\|, \tag{3.1}$$

for each $y, z \in V$.

The set of all locally Lipschitz mappings defined on X with real values is denoted by $\mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$.

Definition 3.2. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x, v \in X$, $v \neq 0$. We call the generalized directional derivative of f in x with respect to the direction v the number

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}. \tag{3.2}$$

We first observe that if f is a locally Lipschitz functional, then $f^0(x, v)$ is a finite number and

$$|f^0(x, v)| \leq k\|v\|. \tag{3.3}$$

Moreover, if $x \in X$ is fixed, then the mapping $v \mapsto f^0(x,v)$ is positive homogeneous and subadditive, so it is convex continuous. By the Hahn-Banach theorem, there exists a linear map $x^* : X \to \mathbb{R}$ such that for every $v \in X$,

$$x^*(v) \le f^0(x,v). \tag{3.4}$$

The continuity of x^* is an immediate consequence of the above inequality and of (3.3).

Definition 3.3. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x \in X$. The generalized gradient (Clarke subdifferential) of f at the point x is the nonempty subset $\partial f(x)$ of X^* which is defined by

$$\partial f(x) = \{x^* \in X^*;\ f^0(x,v) \ge \langle x^*, v\rangle,\ \forall v \in X\}. \tag{3.5}$$

We point out that if f is convex then $\partial f(x)$ coincides with the subdifferential of f in x in the sense of the convex analysis, that is,

$$\partial f(x) = \{x^* \in X^*;\ f(y) - f(x) \ge \langle x^*, y - x\rangle,\ \forall y \in X\}. \tag{3.6}$$

We list in what follows the main properties of the Clarke gradient of a locally Lipschitz functional. We refer to [41, 46, 47] for further details and proofs.

(a) For every $x \in X$, $\partial f(x)$ is a convex and $\sigma(X^*, X)$-compact set.

(b) For every $x, v \in X$, $v \neq 0$, the following holds:

$$f^0(x,v) = \max\{\langle x^*, v\rangle;\ x^* \in \partial f(x)\}. \tag{3.7}$$

(c) The multivalued mapping $x \mapsto \partial f(x)$ is upper semicontinuous, in the sense that for every $x_0 \in X$, $\varepsilon > 0$ and $v \in X$ there exists $\delta > 0$ such that, for any $x^* \in \partial f(x)$ satisfying $\|x - x_0\| < \delta$, there is some $x_0^* \in \partial f(x_0)$ satisfying $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

(d) The functional $f^0(\cdot,\cdot)$ is upper semicontinuous.

(e) If x is an extremum point of f, then $0 \in \partial f(x)$.

(f) The mapping

$$\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\| \tag{3.8}$$

exists and is lower semicontinuous.

(g) $\partial(-f)(x) = -\partial f(x)$.

(h) Lebourg's mean value theorem (see [132]): if x and y are two distinct point in X then there exists a point z situated on the open segment joining x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x\rangle. \tag{3.9}$$

(i) If f has a Gâteaux derivative f' which is continuous in a neighborhood of x, then $\partial f(x) = \{f'(x)\}$. If X is finite dimensional, then $\partial f(x)$ reduces at one point if and only if f is Fréchet-differentiable at x.

Definition 3.4. A point $x \in X$ is said to be a critical point of the locally Lipschitz functional $f : X \to \mathbb{R}$ if $0 \in \partial f(x)$, that is, $f^0(x, v) \geq 0$, for every $v \in X \setminus \{0\}$. A number c is a critical value of f provided that there exists a critical point $x \in X$ such that $f(x) = c$.

Remark that a minimum point is a critical point. Indeed, if x is a local minimum point, then for every $v \in X$,

$$0 \leq \limsup_{\lambda \searrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda} \leq f^0(x, v). \tag{3.10}$$

We now introduce a compactness condition for locally Lipschitz functionals. This condition was used for the first time, in the case of C^1-functionals, by R. Palais and S. Smale (in the global variant) and by H. Brezis, J. M. Coron and L. Nirenberg (in its local variant).

Definition 3.5. If $f : X \to \mathbb{R}$ is a locally Lipschitz functional and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short, $(\mathrm{PS})_c$) if any sequence (x_n) in X satisfying $f(x_n) \to c$ and $\lambda(x_n) \to 0$, contains a convergent subsequence. The mapping f satisfies the Palais-Smale condition (in short, (PS)) if every sequence (x_n) which satisfies that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, has a convergent subsequence.

3.2. Ekeland's variational principle

The following basic theorem is due to Ekeland [75].

Theorem 3.6. *Let (M, d) be a complete metric space and assume that $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$, is a lower semicontinuous function which is bounded from below.*

Then the following properties hold true: for every $\varepsilon > 0$ and for any $z_0 \in M$ there exists $z \in M$ such that

(i) $\psi(z) \leq \psi(z_0) - \varepsilon d(z, z_0)$,
(ii) $\psi(x) \geq \psi(z) - \varepsilon d(x, z)$, *for any $x \in M$.*

Proof. We may assume without loss of generality that $\varepsilon = 1$. Define the following binary relation on M:

$$y \leq x \quad \text{if and only if } \psi(y) - \psi(x) + d(x, y) \leq 0. \tag{3.11}$$

We verify that "$\leq$" is an order relation, that is,

(a) $x \leq x$, for any $x \in M$;
(b) if $x \leq y$ and $y \leq x$ then $x = y$;
(c) if $x \leq y$ and $y \leq z$ then $x \leq z$.

For arbitrary $x \in M$, set

$$S(x) = \{y \in M;\ y \leq x\}. \tag{3.12}$$

Let (ε_n) be a sequence of positive numbers such that $\varepsilon_n \to 0$ and fix $z_0 \in M$. For any $n \geq 0$, let $z_{n+1} \in S(z_n)$ be such that

$$\psi(z_{n+1}) \leq \inf_{S(z_n)} \psi + \varepsilon_{n+1}. \tag{3.13}$$

The existence of z_{n+1} follows by the definition of the set $S(x)$. We prove that the sequence (z_n) converges to some element z which satisfies (i) and (ii).

Let us first remark that $S(y) \subset S(x)$, provided that $y \leq x$. Hence, $S(z_{n+1}) \subset S(z_n)$. It follows that, for any $n \geq 0$,

$$\psi(z_{n+1}) - \psi(z_n) + d(z_n, z_{n+1}) \leq 0, \tag{3.14}$$

which implies $\psi(z_{n+1}) \leq \psi(z_n)$. Since ψ is bounded from below, it follows that the sequence $\{\psi(z_n)\}$ converges.

We prove in what follows that (z_n) is a Cauchy sequence. Indeed, for any n and p we have

$$\psi(z_{n+p}) - \psi(z_n) + d(z_{n+p}, z_n) \leq 0. \tag{3.15}$$

Therefore,

$$d(z_{n+p}, z_n) \leq \psi(z_n) - \psi(z_{n+p}) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.16}$$

which shows that (z_n) is a Cauchy sequence, so it converges to some $z \in M$. Now, taking $n = 0$ in (3.15), we find

$$\psi(z_p) - \psi(z_0) + d(z_p, z_0) \leq 0. \tag{3.17}$$

So, as $p \to \infty$, we find (i).

In order to prove (ii), let us choose an arbitrary $x \in M$. We distinguish the following situations.

Case 1. $x \in S(z_n)$, for any $n \geq 0$. It follows that $\psi(z_{n+1}) \leq \psi(x) + \varepsilon_{n+1}$ which implies that $\psi(z) \leq \psi(x)$.

Case 2. There exists an integer $N \geq 1$ such that $x \notin S(z_n)$, for any $n \geq N$ or, equivalently,

$$\psi(x) - \psi(z_n) + d(x, z_n) > 0, \quad \text{for every } n \geq N. \tag{3.18}$$

Passing at the limit in this inequality as $n \to \infty$ we find (ii). □

Corollary 3.7. *Assume the same hypotheses on M and ψ. Then, for any $\varepsilon > 0$, there exists $z \in M$ such that*

$$\begin{gathered} \psi(z) < \inf_M \psi + \varepsilon, \\ \psi(x) \geq \psi(z) - \varepsilon\, d(x, z), \quad \textit{for any } x \in M. \end{gathered} \tag{3.19}$$

The conclusion follows directly from Theorem 3.6.

The following consequence of Ekeland's variational principle is of particular interest in our next arguments. Roughly speaking, this property establishes the existence of *almost critical points* for bounded from below C^1-functionals defined on Banach spaces. In order words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

Corollary 3.8. *Let E be a Banach space and let $\psi : E \to \mathbb{R}$ be a C^1 function which is bounded from below. Then, for any $\varepsilon > 0$, there exists $z \in E$ such that*

$$\psi(z) \le \inf_E \psi + \varepsilon, \qquad ||\psi'(z)||_{E^\star} \le \varepsilon. \tag{3.20}$$

Proof. The first part of the conclusion follows directly from Theorem 3.6. For the second part we have

$$||\psi'(z)||_{E^\star} = \sup_{\|u\|=1} \langle \psi'(z), u \rangle. \tag{3.21}$$

But

$$\langle \psi'(z), u \rangle = \lim_{\delta \to 0} \frac{\psi(z+\delta u) - \psi(z)}{\delta \|u\|}. \tag{3.22}$$

So, by Theorem 3.6,

$$\langle \psi'(z), u \rangle \ge -\varepsilon. \tag{3.23}$$

Replacing now u by $-u$ we find

$$\langle \psi'(z), u \rangle \le \varepsilon, \tag{3.24}$$

which concludes our proof. □

We give in what follows a variant of Ekeland's variational principle in the case of *finite dimensional* Banach spaces.

Theorem 3.9. *Let $\psi : \mathbb{R}^N \to (-\infty, +\infty]$ be a lower semicontinuous function, $\psi \not\equiv +\infty$, bounded from below. Let $x_\varepsilon \in \mathbb{R}^N$ be such that*

$$\inf \psi \le \psi(x_\varepsilon) \le \inf \psi + \varepsilon. \tag{3.25}$$

Then, for every $\lambda > 0$, there exists $z_\varepsilon \in \mathbb{R}^N$ such that

(i) $\psi(z_\varepsilon) \le \psi(x_\varepsilon)$,
(ii) $\|z_\varepsilon - x_\varepsilon\| \le \lambda$,
(iii) $\psi(z_\varepsilon) \le \psi(x) + (\varepsilon/\lambda)\|z_\varepsilon - x\|$, *for every $x \in \mathbb{R}^N$.*

Proof. Given x_ε satisfying (3.25), let us consider the function $\varphi : \mathbb{R}^N \to (-\infty, +\infty]$ defined by

$$\varphi(x) = \psi(x) + \frac{\varepsilon}{\lambda}||x - x_\varepsilon||. \tag{3.26}$$

By our hypotheses on ψ it follows that φ is lower semicontinuous and bounded from below. Moreover, $\varphi(x) \to +\infty$ as $\|x\| \to +\infty$. Therefore, there exists $z_\varepsilon \in \mathbb{R}^N$ which minimizes φ on $\mathbb{R}^N$, that is, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) + \frac{\varepsilon}{\lambda}||z_\varepsilon - x_\varepsilon|| \le \psi(x) + \frac{\varepsilon}{\lambda}||x - x_\varepsilon||. \tag{3.27}$$

By letting $x = x_\varepsilon$, we find

$$\psi(z_\varepsilon) + \frac{\varepsilon}{\lambda}||z_\varepsilon - x_\varepsilon|| \le \psi(x_\varepsilon), \tag{3.28}$$

and (i) follows. Next, since $\psi(x_\varepsilon) \le \inf \psi + \varepsilon$, we deduce from the above inequality that $\|z_\varepsilon - x_\varepsilon\| \le \lambda$.

We infer from (3.27) that, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) \le \psi(x) + \frac{\varepsilon}{\lambda}(||x - x_\varepsilon|| - ||z_\varepsilon - x_\varepsilon||) \le \psi(x) + \frac{\varepsilon}{\lambda}||x - z_\varepsilon|| \tag{3.29}$$

which is exactly the desired inequality (iii). □

The above result shows that, the closer to x_ε we desire z_ε to be, the larger the perturbation of ψ that must be accepted. In practise, a good compromise is to take $\lambda = \sqrt{\varepsilon}$.

It is striking to remark that the Ekeland variational principle characterizes the completeness of a metric space in a certain sense. More precisely, we have the following property.

Theorem 3.10. *Let (M, d) be a metric space. Then M is complete if and only if the following holds: for every application $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$, which is bounded from below and for every $\varepsilon > 0$, there exists $z_\varepsilon \in M$ such that*

(i) $\psi(z_\varepsilon) \le \inf_M \psi + \varepsilon$,
(ii) $\psi(z) > \psi(z_\varepsilon) - \varepsilon d(z, z_\varepsilon)$, *for any $z \in M \setminus \{z_\varepsilon\}$.*

Proof. The "only if" part follows directly from Corollary 3.7.

For the converse, let us assume that M is an arbitrary metric space satisfying the hypotheses. Let $(v_n) \subset M$ be an arbitrary Cauchy sequence and consider the function $f : M \to \mathbb{R}$ defined by

$$f(u) = \lim_{n\to\infty} d(u, v_n). \tag{3.30}$$

The function f is continuous and, since (v_n) is a Cauchy sequence, then $\inf f = 0$. In order to justify the completeness of M it is enough to find $v \in M$ such that $f(v) = 0$. For this aim, choose arbitrarily $\varepsilon \in (0, 1)$. Now, from our hypotheses (i) and (ii), there exists $v \in M$ such that $f(v) \le \varepsilon$ and

$$f(w) + \varepsilon d(w, v) > f(v), \quad \text{for any } w \in M \setminus \{v\}. \tag{3.31}$$

From the definition of f and the fact that (v_n) is a Cauchy sequence, we can take $w = v_k$ for k large enough such that $f(w)$ is arbitrarily small and $d(w, v) \le \varepsilon + \eta$, for any $\eta > 0$, because $f(v) \le \varepsilon$. Using (ii) we obtain that, in fact, $f(v) \le \varepsilon^k$. Repeating the argument we may conclude that $f(v) \le \varepsilon^n$, for all $n \ge 1$ and so $f(v) = 0$, as required. □

3.3. Nonsmooth mountain pass type theorems

As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

Arthur Cayley (1821–1895)

The mountain pass theorem of Ambrosetti and Rabinowitz [14] is a result of great intuitive appeal which is very useful to find the critical points of functionals, particularly those that occur in the theory of ordinary and partial differential equations. In its main form, as given originally by A. Ambrosetti and P. Rabinowitz, the mountain pass theorem considers a function $J : X \to \mathbb{R}$ of class C^1 satisfying the following geometric conditions:

(a) there exist two numbers $R > 0$ and a such that $J(u) \geq a$ for every $u \in X$ with $\|u\| = R$;
(b) $J(0) < a$ and $J(v) < a$ for some $v \in X$ with $\|v\| > R$.

With an additional compactness condition of Palais-Smale type it then follows that the functional J has a critical point $u_0 \in X \setminus \{0, v\}$ with corresponding critical value $c \geq a$. Roughly speaking, this critical value occurs because the *villages* 0 and v are low points on either side of the *mountain* $S_R := \{u \in X;\ \|u\| = R\}$, so that between 0 and v there should exist a lowest critical point, the so-called *mountain pass*.

We give in what follows a generalization of the mountain pass theorem in the framework of locally Lipschitz functionals.

Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional. Consider K a compact metric space and K^* a closed nonempty subset of K. If $p^* : K^* \to X$ is a continuous mapping, set

$$\mathcal{P} = \{p \in C(K, X);\ p = p^* \text{ on } K^*\}. \tag{3.32}$$

By a celebrated theorem of Dugundji [70], the set $\mathcal{P}$ is nonempty.

Define

$$c = \inf_{p \in \mathcal{P}} \max_{t \in K} f\big(p(t)\big). \tag{3.33}$$

Obviously, $c \geq \max_{t \in K^*} f(p^*(t))$.

Theorem 3.11. *Assume that*

$$c > \max_{t \in K^*} f\big(p^*(t)\big). \tag{3.34}$$

Then there exists a sequence (x_n) in X such that

(i) $\lim_{n \to \infty} f(x_n) = c$;
(ii) $\lim_{n \to \infty} \lambda(x_n) = 0$.

For the proof of this theorem, we need the following auxiliary result.

Lemma 3.12. *Let M be a compact metric space and let $\varphi : M \to 2^{X^*}$ be an upper semicontinuous mapping such that, for every $t \in M$, the set $\varphi(t)$ is convex and $\sigma(X^*, X)$-compact. For $t \in M$, denote*

$$\gamma(t) = \inf\{\|x^*\|;\ x^* \in \varphi(t)\},\qquad \gamma = \inf_{t\in M} \gamma(t). \tag{3.35}$$

Then, for every fixed $\varepsilon > 0$, there exists a continuous mapping $v : M \to X$ such that for every $t \in M$ and $x^* \in \varphi(t)$,

$$\|v(t)\| \le 1, \qquad \langle x^*, v(t)\rangle \ge \gamma - \varepsilon. \tag{3.36}$$

Proof of Lemma 3.12. Assume, without loss of generality, that $\gamma > 0$ and $0 < \varepsilon < \gamma$. Denoting by B_r the open ball in X^* centered at the origin and with radius r then, for every $t \in M$ we have

$$B_{\gamma-\varepsilon/2} \cap \varphi(t) = \emptyset. \tag{3.37}$$

Since $\varphi(t)$ and $B_{\gamma-\epsilon/2}$ are convex, disjoint, and $\sigma(X^*, X)$-compact sets, it follows from the separation theorem in locally convex spaces [204, Theorem 3.4], applied to the space $(X^*, \sigma(X^*, X))$, and from the fact that the dual of this space is X, that for every $t \in M$, there exists $v_t \in X$ such that

$$\|v_t\| = 1, \langle \xi, v_t\rangle \le \langle x^*, v_t\rangle, \tag{3.38}$$

for any $\xi \in B_{\gamma-\epsilon/2}$ and for every $x^* \in \varphi(t)$.

Hence, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t\rangle \ge \sup_{\xi\in B_{\gamma-\varepsilon/2}} \langle \xi, v_t\rangle = \gamma - \frac{\varepsilon}{2}. \tag{3.39}$$

Since φ is upper semicontinuous, there exists an open neighborhood $V(t)$ of t such that for every $t' \in V(t)$ and all $x^* \in \varphi(t')$,

$$\langle x^*, v_t\rangle > \gamma - \varepsilon. \tag{3.40}$$

Therefore, since M is compact and $M = \bigcup_{t\in M} V(t)$, there exists an open covering $\{V_1, \ldots, V_n\}$ of M. Let $v_1, \ldots, v_n$ be on the unit sphere of X such that

$$\langle x^*, v_i\rangle > \gamma - \varepsilon, \tag{3.41}$$

for every $1 \le i \le n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \operatorname{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum_{j=1}^n \rho_j(t)}, \qquad v(t) = \sum_{i=1}^n \zeta_i(t) v_i. \tag{3.42}$$

A straightforward computation shows that v satisfies our conclusion. □

Proof of Theorem 3.11. We apply Ekeland's variational principle to the functional

$$\psi(p) = \max_{t\in K} f(p(t)), \tag{3.43}$$

defined on $\mathcal{P}$, which is a complete metric space if it is endowed with the metric

$$d(p,q) = \max_{t\in K} \|p(t) - q(t)\|, \quad \text{for any } p,q \in \mathcal{P}. \tag{3.44}$$

The mapping ψ is continuous and bounded from below because, for every $p \in \mathcal{P}$,

$$\psi(p) \geq \max_{t\in K^*} f(p^*(t)). \tag{3.45}$$

Since

$$c = \inf_{p\in\mathcal{P}} \psi(p), \tag{3.46}$$

it follows that for every $\varepsilon > 0$, there is some $p \in \mathcal{P}$ such that

$$\begin{gathered} \psi(q) - \psi(p) + \varepsilon d(p,q) \geq 0 \quad \forall q \in \mathcal{P}, \\ c \leq \psi(p) \leq c + \varepsilon. \end{gathered} \tag{3.47}$$

Set

$$B(p) = \{t \in K;\ f(p(t)) = \psi(p)\}. \tag{3.48}$$

For concluding the proof, it is sufficient to show that there exists $t' \in B(p)$ such that

$$\lambda(p(t')) \leq 2\varepsilon. \tag{3.49}$$

Indeed, the conclusion of the theorem follows then easily by choosing $\varepsilon = 1/n$ and $x_n = p(t')$. Applying Lemma 3.12 for $M = B(p)$ and $\varphi(t) = \partial f(p(t))$, we obtain a continuous map $v : B(p) \to X$ such that for every $t \in B(p)$ and $x^* \in \partial f(p(t))$ we have

$$\|v(t)\| \leq 1, \qquad \langle x^*, v(t)\rangle \geq \gamma - \varepsilon, \tag{3.50}$$

where

$$\gamma = \inf_{t\in B(p)} \lambda(p(t)). \tag{3.51}$$

It follows that for every $t \in B(p)$,

$$\begin{aligned} f^0(p(t), -v(t)) &= \max\{\langle x^*, -v(t)\rangle;\ x^* \in \partial f(p(t))\} \\ &= -\min\{\langle x^*, v(t);\ x^* \in \partial f(p(t))\} \leq -\gamma + \varepsilon. \end{aligned} \tag{3.52}$$

By (3.34) we have $B(p) \cap K^* = \emptyset$. So, there exists a continuous extension $w : K \to X$ of v such that $w = 0$ pe K^* and, for every $t \in K$,

$$\|w(t)\| \leq 1. \tag{3.53}$$

Choose in the place of q in (3.47) small perturbations of the path p:

$$q_h(t) = p(t) - hw(t), \tag{3.54}$$

where $h > 0$ is small enough.

We deduce from (3.47) that, for every $h > 0$,

$$-\varepsilon \leq -\varepsilon \|w\|_\infty \leq \frac{\psi(q_h) - \psi(p)}{h}. \tag{3.55}$$

In what follows, $\varepsilon > 0$ will be fixed, while $h \to 0$. Let $t_h \in K$ be such that $f(q_h(t_h)) = \psi(q_h)$. We may also assume that the sequence (t_{h_n}) converges to some t_0, which, obviously, is in $B(p)$. Observe that

$$\frac{\psi(q_h) - \psi(p)}{h} = \frac{\psi(p - hw) - \psi(p)}{h} \leq \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}. \tag{3.56}$$

It follows from this relation and from (3.55) that

$$\begin{aligned} -\varepsilon &\leq \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h} \\ &\leq \frac{f(p(t_h) - hw(t_0)) - f(p(t_h))}{h} + \frac{f(p(t_h) - hw(t_h)) - f(p(t_h) - hw(t_0))}{h}. \end{aligned} \tag{3.57}$$

Using the fact that f is a locally Lipschitz map and $t_{h_n} \to t_0$, we find that

$$\lim_{n\to\infty} \frac{f(p(t_{h_n}) - h_n w(t_{h_n})) - f(p(t_{h_n}) - h_n w(t_0))}{h_n} = 0. \tag{3.58}$$

Therefore,

$$-\varepsilon \leq \limsup_{n\to\infty} \frac{f(p(t_0) + z_n - h_n w(t_0)) - f(p(t_0) + z_n)}{h_n}, \tag{3.59}$$

where $z_n = p(t_{h_n}) - p(t_0)$. Consequently,

$$-\varepsilon \leq f^0(p(t_0), -w(t_0)) = f^0(p(t_0), -v(t_0)) \leq -\gamma + \varepsilon. \tag{3.60}$$

It follows that

$$\gamma = \inf\{\|x^*\|;\ x^* \in \partial f(p(t)), t \in B(p)\} \leq 2\varepsilon. \tag{3.61}$$

Taking into account the lower semicontinuity of λ, we deduce the existence of some $t' \in B(p)$ such that

$$\lambda(p(t')) = \inf\{\|x^*\|;\ x^* \in \partial f(p(t'))\} \le 2\varepsilon. \tag{3.62}$$

This concludes the proof. □

Corollary 3.13. *If f satisfies the condition $(\mathrm{PS})_c$ and the hypotheses of Theorem 3.11, then c is a critical value of f corresponding to a critical point which is not in $p^*(K^*)$.*

Proof. The proof of this result follows easily by applying Theorem 3.11 and the fact that λ is lower semicontinuous. □

The following result generalizes the classical mountain pass theorem of A. Ambrosetti and P. Rabinowitz.

Corollary 3.14. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional such that $f(0) = 0$ and there exists $v \in X \setminus \{0\}$ so that $f(v) \le 0$. Set*

$$\begin{gathered} \mathcal{P} = \{p \in C([0,1],X);\ p(0) = 0,\ p(1) = v\}, \\ c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t)). \end{gathered} \tag{3.63}$$

If $c > 0$ and f satisfies the condition $(\mathrm{PS})_c$, then c is a critical value of f.

Proof. For the proof of this result, it is sufficient to apply Corollary 3.13 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = 0$, and $p^*(1) = v$. □

Corollary 3.15. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz mapping. Assume that there exists a subset S of X such that, for every $p \in \mathcal{P}$,*

$$p(K) \cap S \ne \emptyset. \tag{3.64}$$

If

$$\inf_{x\in S} f(x) > \max_{t\in K^*} f(p^*(t)), \tag{3.65}$$

then the conclusion of Theorem 3.11 holds.

Proof. It suffices to observe that

$$\inf_{p\in\mathcal{P}} \max_{t\in K} f(p(t)) \ge \inf_{x\in S} f(x) > \max_{t\in K^*} f(p^*(t)). \tag{3.66}$$

Then our conclusion follows directly. □

Using now Theorem 3.11, we may prove the following result, which is originally due to Brezis, Coron, and Nirenberg (see [31, Theorem 2]).

Corollary 3.16. *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable functional such that $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. If f satisfies* (3.34)*, then there exists a sequence (x_n) in X such that*

(i) $\lim_{n\to\infty} f(x_n) = c$,

(ii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.

Moreover, if f satisfies the condition $(\mathrm{PS})_c$*, then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

Proof. Observe first that f' is locally bounded. Indeed, if (x_n) is a sequence converging to x_0, then

$$\sup_n |\langle f'(x_n), v\rangle| < \infty, \tag{3.67}$$

for every $v \in X$. Thus, by the Banach-Steinhaus theorem,

$$\limsup_{n\to\infty} ||f'(x_n)|| < \infty. \tag{3.68}$$

For $\lambda > 0$ small enough and $h \in X$ sufficiently small, we have

$$|f(x_0 + h + \lambda v) - f(x_0 + h)| = |\lambda\langle f'(x_0 + h + \lambda\theta v), v\rangle| \le C\|\lambda v\|, \tag{3.69}$$

where $\theta \in (0,1)$. Therefore, $f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ and $f^0(x_0, v) = \langle f'(x_0), v\rangle$, by the continuity assumption on f'. In relation (3.69), the existence of C follows from the local boundedness property of f'.

Since f^0 is linear in v, we obtain

$$\partial f(x) = \{f'(x)\}. \tag{3.70}$$

To conclude the proof, it remains to apply Theorem 3.11 and Corollary 3.13. □

A very useful result in applications is the following variant of the *saddle point* theorem of P. Rabinowitz.

Corollary 3.17. *Assume that X admits a decomposition of the form $X = X_1 \oplus X_2$, where X_2 is a finite dimensional subspace of X. For some fixed $z \in X_2$, suppose that there exists $R > \|z\|$ such that*

$$\inf_{x\in X_1} f(x+z) > \max_{x\in K^*} f(x), \tag{3.71}$$

where

$$K^* = \{x \in X_2;\ \|x\| = R\}. \tag{3.72}$$

Set

$$\begin{aligned} K &= \{x \in X_2;\ \|x\| \le R\},\\ \mathcal{P} &= \{p \in C(K, X);\ p(x) = x \text{ if } \|x\| = R\}. \end{aligned} \tag{3.73}$$

If c is chosen as in (3.33) *and f satisfies the condition* $(\mathrm{PS})_c$*, then c is a critical value of f.*

Proof. Applying Corollary 3.15 for $S = z + X_1$, we observe that it is sufficient to prove that, for every $p \in \mathcal{P}$,

$$p(K) \cap (z + X_1) \neq \emptyset. \tag{3.74}$$

If $P : X \to X_2$ is the canonical projection, then the above condition is equivalent to the fact that for every $p \in \mathcal{P}$, there exists $x \in K$ such that

$$P\big(p(x) - z\big) = P\big(p(x)\big) - z = 0. \tag{3.75}$$

To prove this claim, we use an argument based on the topological degree theory. Indeed, for every fixed $p \in \mathcal{P}$ we have

$$P \circ p = \mathrm{Id} \ \text{ on } K^* = \partial K. \tag{3.76}$$

Hence

$$d(P \circ p - z, \operatorname{Int} K, 0) = d(P \circ p, \operatorname{Int} K, z) = d(\mathrm{Id}, \operatorname{Int} K, z) = 1. \tag{3.77}$$

Now, by the existence property of the Brouwer degree, we may find $x \in \operatorname{Int} K$ such that

$$(P \circ p)(x) - z = 0, \tag{3.78}$$

which concludes our proof. □

3.3.1. Mountains of zero altitude

It is natural to ask us what happens if the condition (3.34) fails to be valid, more precisely, if

$$c = \max_{t \in K^*} f\big(p^*(t)\big). \tag{3.79}$$

The following example shows that in this case the conclusion of Theorem 3.11 does not hold.

Example 3.18. Let $X = \mathbb{R}^2$, $K = [0,1] \times \{0\}$, $K^* = \{(0,0),(1,0)\}$ and let p^* be the identic map of K^*. As locally Lipschitz functional we choose

$$f : X \longrightarrow \mathbb{R}, \quad f(x,y) = x + |y|. \tag{3.80}$$

In this case,

$$c = \max_{t \in K^*} f\big(p^*(t)\big) = 1. \tag{3.81}$$

An elementary computation shows that

$$\partial f(x,y) = \begin{pmatrix}1\\1\end{pmatrix} \quad \text{if } y > 0,$$

$$\partial f(x,y) = \begin{pmatrix}1\\-1\end{pmatrix} \quad \text{if } y < 0, \tag{3.82}$$

$$\partial f(x,0) = \left\{\begin{pmatrix}1\\a\end{pmatrix} \text{ if } -1 \le a \le 1\right\}.$$

It follows that f satisfies the Palais-Smale condition. However, f has no critical point.

In the smooth framework, the mountain pass theorem in the *zero altitude* case was proved by Pucci and Serrin [180]. They consider a function $J : X \to \mathbb{R}$ of class C^1 satisfying the following geometric conditions:

(a) there exist real numbers a, r, R such that $0 < r < R$ and $J(u) \ge a$ for every $u \in X$ with $r < \|u\| < R$;
(b) $J(0) \le a$ and $J(v) \le a$ for some $v \in X$ with $\|v\| > R$.

Under these hypotheses, combined with the standard Palais-Smale compactness condition, Pucci and Serrin established the existence of a critical point $u_0 \in X \setminus \{0, v\}$ of J with corresponding critical value $c \ge a$. Moreover, if $c = a$ then the critical point can be chosen with $r < \|u_0\| < R$. Roughly speaking, the mountain pass theorem continues to hold for a mountain of zero altitude, provided it also has nonzero thickness; in addition, if $c = a$, then the pass itself occurs precisely on the mountain, in the sense that it satisfies $r < \|u_0\| < R$.

The following result gives a sufficient condition so that Theorem 3.11 holds provided that condition (3.34) fails.

Theorem 3.19. *Assume that for every $p \in \mathcal{P}$ there exists $t \in K \setminus K^*$ such that $f(p(t)) \ge c$. Then there exists a sequence (x_n) in X such that*

(i) $\lim_{n\to\infty} f(x_n) = c$;
(ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Moreover, if f satisfies the condition $(\mathrm{PS})_c$, then c is a critical value of f. Furthermore, if (p_n) is an arbitrary sequence in $\mathcal{P}$ satisfying

$$\lim_{n\to\infty} \max_{t\in K} f(p_n(t)) = c, \tag{3.83}$$

then there exists a sequence (t_n) in K such that

$$\lim_{n\to\infty} f(p_n(t_n)) = c, \qquad \lim_{n\to\infty} \lambda(p_n(t_n)) = 0. \tag{3.84}$$

Proof. For every $\varepsilon > 0$ we apply Ekeland's variational principle to the perturbed functional

$$\psi_\varepsilon : \mathcal{P} \longrightarrow \mathbb{R}, \quad \psi_\varepsilon(p) = \max_{t\in K} \left(f(p(t)) + \varepsilon d(t)\right), \tag{3.85}$$

where

$$d(t) = \min\{\operatorname{dist}(t, K^*), 1\}. \tag{3.86}$$

If

$$c_\varepsilon = \inf_{p \in \mathcal{P}} \psi_\varepsilon(p), \tag{3.87}$$

then

$$c \leq c_\varepsilon \leq c + \varepsilon. \tag{3.88}$$

Thus, by Ekeland's variational principle, there exists a path $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$,

$$\begin{gathered} \psi_\varepsilon(q) - \psi_\varepsilon(p) + \varepsilon d(p,q) \geq 0, \\ c \leq c_\varepsilon \leq \psi_\varepsilon(p) \leq c_\varepsilon + \varepsilon \leq c + 2\varepsilon. \end{gathered} \tag{3.89}$$

Denoting

$$B_\varepsilon(p) = \{t \in K;\ f(p(t)) + \varepsilon d(t) = \psi_\varepsilon(p)\}, \tag{3.90}$$

it remains to show that there is some $t' \in B_\varepsilon(p)$ such that $\lambda(p(t')) \leq 2\varepsilon$. Indeed, the conclusion of the first part of the theorem follows easily, by choosing $\varepsilon = 1/n$ and $x_n = p(t')$.

Now, by Lemma 3.12 applied for $M = B_\varepsilon(p)$ and $\varphi(t) = \partial f(p(t))$, we find a continuous mapping $v : B_\varepsilon(p) \to X$ such that, for every $t \in B_\varepsilon(p)$ and all $x^* \in \partial f(p(t))$,

$$\|v(t)\| \leq 1, \qquad \langle x^*, v(t)\rangle \geq \gamma_\varepsilon - \varepsilon, \tag{3.91}$$

where

$$\gamma_\varepsilon = \inf_{t \in B_\varepsilon(p)} \lambda(p(t)). \tag{3.92}$$

On the other hand, it follows by our hypothesis that

$$\psi_\varepsilon(p) > \max_{t \in K^*} f(p(t)). \tag{3.93}$$

Hence

$$B_\varepsilon(p) \cap K^* = \emptyset. \tag{3.94}$$

So, there exists a continuous extension w of v, defined on K and such that

$$w = 0 \text{ on } K^*, \qquad \|w(t)\| \leq 1, \quad \text{for any } t \in K. \tag{3.95}$$

Choose as paths q in relation (3.89) small variations of the path p:

$$q_h(t) = p(t) - hw(t), \tag{3.96}$$

for $h > 0$ sufficiently small.

In what follows $\varepsilon > 0$ will be fixed, while $h \to 0$.

Let $t_h \in B_\varepsilon(p)$ be such that

$$f(q(t_h)) + \varepsilon d(t_h) = \psi_\varepsilon(q_h). \tag{3.97}$$

There exists a sequence (h_n) converging to 0 and such that the corresponding sequence (t_{h_n}) converges to some t_0, which, obviously, lies in $B_\varepsilon(p)$. It follows that

$$\begin{aligned}
-\varepsilon \le -\varepsilon\|w\|_\infty
&\le \frac{\psi_\varepsilon(q_h) - \psi_\varepsilon(p)}{h}\\
&= \frac{f(q_h(t_h)) + \varepsilon d(t_h) - \psi_\varepsilon(p)}{h}\\
&\le \frac{f(q_h(t_h)) - f(p(t_h))}{h}\\
&= \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}.
\end{aligned} \tag{3.98}$$

With the same arguments as in the proof of Theorem 3.11 we obtain the existence of some $t' \in B_\varepsilon(p)$ such that

$$\lambda(p(t')) \le 2\varepsilon. \tag{3.99}$$

Furthermore, if f satisfies $(\mathrm{PS})_c$ then c is a critical value of f, since λ is lower semicontinuous.

For the second part of the proof, applying again Ekeland's variational principle, we deduce the existence of a sequence of paths (q_n) in $\mathcal{P}$ such that, for every $q \in \mathcal{P}$,

$$\begin{gathered}
\psi_{\varepsilon_n^2}(q) - \psi_{\varepsilon_n^2}(q_n) + \varepsilon_n d(q, q_n) \ge 0,\\
\psi_{\varepsilon_n^2}(q_n) \le \psi_{\varepsilon_n^2}(p_n) - \varepsilon_n d(p_n, q_n),
\end{gathered} \tag{3.100}$$

where (ε_n) is a sequence of positive numbers converging to 0 and (p_n) are paths in $\mathcal{P}$ such that

$$\psi_{\varepsilon_n^2}(p_n) \le c + 2\varepsilon_n^2. \tag{3.101}$$

Applying the same argument for q_n, instead of p, we find $t_n \in K$ such that

$$\begin{gathered}
c - \varepsilon_n^2 \le f(q_n(t_n)) \le c + 2\varepsilon_n^2,\\
\lambda(q_n(t_n)) \le 2\varepsilon_n.
\end{gathered} \tag{3.102}$$

We argue that this is the desired sequence (t_n). Indeed, by the Palais-Smale condition $(\mathrm{PS})_c$, there exists a subsequence of $(q_n(t_n))$ which converges to a critical point. The corresponding subsequence of $(p_n(t_n))$ converges to the same limit. A standard argument, based on the continuity of f and the lower semicontinuity of λ, shows that for all the sequence we have

$$\lim_{n\to\infty} f(p_n(t_n)) = c,$$
$$\lim_{n\to\infty} \lambda(p_n(t_n)) = 0. \tag{3.103}$$

This concludes our proof. □

Corollary 3.20. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional which satisfies the Palais-Smale condition.*

If f has two different minimum points, then f possesses a third critical point.

Proof. Let x_0 and x_1 be two different minimum points of f.

Case 1 ($f(x_0) = f(x_1) = a$). Choose $0 < R < (1/2)\|x_1 - x_0\|$ such that $f(x) \geq a$, for all $x \in B(x_0, R) \cup B(x_1, R)$.

Set $A = \overline{B}(x_0, R/2) \cup \overline{B}(x_1, R/2)$.

Case 2 ($f(x_0) > f(x_1)$). Choose $0 < R < \|x_1 - x_0\|$ such that $f(x) \geq f(x_0)$, for every $x \in B(x_0, R)$. Put $A = \overline{B}(x_0, R/2) \cup \{x_1\}$.

In both cases, fix $p^* \in C([0,1], X)$ such that $p^*(0) = x_0$ and $p^*(1) = x_1$. If $K^* = (p^*)^{-1}(A)$ then, by Theorem 3.19, we obtain the existence of a critical point of f, which is different from x_0 and x_1, as we can easily deduce by examining the proof of Theorem 3.19. □

With the same proof as of Corollary 3.16 one can deduce the following mountain pass property which extends the Pucci and Serrin theorem [180, Theorem 1].

Corollary 3.21. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a Gâteaux-differentiable functional such that the operator $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Assume that for every $p \in \mathcal{P}$ there exists $t \in K \setminus K^*$ such that $f(p(t)) \geq c$.*

Then there exists a sequence (x_n) in X so that

(i) $\lim_{n\to\infty} f(x_n) = c$,
(ii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.

If, furthermore, f satisfies $(\mathrm{PS})_c$, *then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

3.3.2. The nonsmooth Ghoussoub-Preiss theorem

The following result is a strengthened variant of Theorems 3.11 and 3.19. The smooth case corresponding to C^1-functionals is due to Ghoussoub and Preiss [98].

Theorem 3.22. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and let F be a closed subset of X, with no common point with $p^*(K^*)$. Assume that*

$$f(x) \geq c, \quad \text{for every } x \in F \tag{3.104}$$

$$p(K) \cap F \neq \emptyset \quad \forall p \in \mathcal{P}. \tag{3.105}$$

Then there exists a sequence (x_n) in X such that
(i) $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0$,
(ii) $\lim_{n\to\infty} f(x_n) = c$,
(iii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Proof. Fix $\varepsilon > 0$ such that

$$\varepsilon < \min\{1; \operatorname{dist}(p^*(K^*), F)\}. \tag{3.106}$$

Choose $p \in \mathcal{P}$ so that

$$\max_{t\in K} f(p(t)) \leq c + \frac{\varepsilon^2}{4}. \tag{3.107}$$

The set

$$K_0 = \{t \in K; \operatorname{dist}(p(t), F) \geq \varepsilon\} \tag{3.108}$$

is bounded and contains K^*. Define

$$\mathcal{P}_0 = \{q \in C(K, X);\ q = p \text{ on } K_0\}. \tag{3.109}$$

Set

$$\eta : X \longrightarrow \mathbb{R}, \qquad \eta(x) = \max\{0; \varepsilon^2 - \varepsilon \operatorname{dist}(x, F)\}. \tag{3.110}$$

Define $\psi : \mathcal{P}_0 \to \mathbb{R}$ by

$$\psi(q) = \max_{t\in K}(f + \eta)(q(t)). \tag{3.111}$$

The functional ψ is continuous and bounded from below. By Ekeland's variational principle, there exists $p_0 \in \mathcal{P}_0$ such that for every $q \in \mathcal{P}_0$,

$$\psi(p_0) \leq \psi(q), \tag{3.112}$$

$$d(p_0, q) \leq \frac{\varepsilon}{2}, \tag{3.113}$$

$$\psi(p_0) \leq \psi(q) + \frac{\varepsilon}{2} d(q, p_0). \tag{3.114}$$

The set

$$B(p_0) = \{t \in K;\ (f + \eta)(p_0(t)) = \psi(p_0)\} \tag{3.115}$$

is closed. To conclude the proof, it is sufficient to show that there exists $t \in B(p_0)$ such that

$$\operatorname{dist}\left(p_0(t), F\right) \leq \frac{3\varepsilon}{2}, \tag{3.116}$$

$$c \leq f\left(p_0(t)\right) \leq c + \frac{5\varepsilon^2}{4}, \tag{3.117}$$

$$\lambda\left(p_0(t)\right) \leq \frac{5\varepsilon}{2}. \tag{3.118}$$

Indeed, it is enough to choose then $\varepsilon = 1/n$ and $x_n = p_0(t)$. □

Proof of (3.116). It follows by the definition of $\mathcal{P}_0$ and (3.105) that, for every $q \in \mathcal{P}_0$, we have

$$q(K \setminus K_0) \cap F \neq \emptyset. \tag{3.119}$$

Therefore, for any $q \in \mathcal{P}_0$,

$$\psi(q) \geq c + \varepsilon^2. \tag{3.120}$$

On the other hand,

$$\psi(p) \leq c + \frac{\varepsilon^2}{4} + \varepsilon^2 = c + \frac{5\varepsilon^2}{4}. \tag{3.121}$$

Hence

$$c + \varepsilon^2 \leq \psi\left(p_0\right) \leq \psi(p) \leq c + \frac{5\varepsilon^2}{4}. \tag{3.122}$$

So, for each $t \in B(p_0)$,

$$c + \varepsilon^2 \leq \psi\left(p_0\right) = (f + \eta)\left(p_0(t)\right). \tag{3.123}$$

Moreover, if $t \in K_0$, then

$$(f + \eta)\left(p_0(t)\right) = (f + \eta)\left(p(t)\right) = f\left(p(t)\right) \leq c + \frac{\varepsilon^2}{4}. \tag{3.124}$$

This implies that

$$B\left(p_0\right) \subset K \setminus K_0. \tag{3.125}$$

By the definition of K_0 it follows that for every $t \in B(p_0)$ we have

$$\operatorname{dist}\left(p(t), F\right) \leq \varepsilon. \tag{3.126}$$

Now, the relation (3.113) yields

$$\operatorname{dist}\left(p_0(t), F\right) \leq \frac{\varepsilon}{2}. \tag{3.127}$$

□

Proof of (3.117). For every $t \in B(p_0)$ we have

$$\psi(p_0) = (f + \eta)(p_0(t)). \tag{3.128}$$

Using (3.122) and taking into account that

$$0 \le \eta \le \varepsilon^2, \tag{3.129}$$

it follows that

$$c \le f(p_0(t)) \le c + \frac{5\varepsilon^2}{4}. \tag{3.130}$$

□

Proof of (3.118). Applying Lemma 3.12 for $\varphi(t) = \partial f(p_0(t))$, we find a continuous mapping $v : B(p_0) \to X$ such that for every $t \in B(p_0)$,

$$||v(t)|| \le 1. \tag{3.131}$$

Moreover, for any $t \in B(p_0)$ and $x^* \in \partial f(p_0(t))$,

$$\langle x^*, v(t) \rangle \ge \gamma - \varepsilon, \tag{3.132}$$

where

$$\gamma = \inf_{t \in B(p_0)} \lambda(p_0(t)). \tag{3.133}$$

Hence for every $t \in B(p_0)$,

$$\begin{aligned} f^0(p_0(t), -v(t)) &= \max\{\langle x^*, -v(t)\rangle; x^* \in \partial f(p_0(t))\} \\ &= -\min\{\langle x^*, v(t)\rangle; x^* \in \partial f(p_0(t))\} \\ &\le -\gamma + \varepsilon. \end{aligned} \tag{3.134}$$

Since $B(p_0) \cap K_0 = \emptyset$, there exists a continuous extension w of v to the set K such that $w = 0$ on K_0 and $\|w(t)\| \le 1$, for all $t \in K$.

Now, by relation (3.114), it follows that for every $\lambda > 0$,

$$-\frac{\varepsilon}{2} \le -\frac{\varepsilon}{2}\|w\|_\infty \le \frac{\psi(p_0 - \lambda w) - \psi(p_0)}{\lambda}. \tag{3.135}$$

For every n, there exists $t_n \in K$ such that

$$\psi\left(p_0 - \frac{1}{n}w\right) = (f + \eta)\left(p_0(t_n) - \frac{1}{n}w(t_n)\right). \tag{3.136}$$

Passing eventually to a subsequence, we may suppose that (t_n) converges to t_0, which, clearly, lies in $B(p_0)$. On the other hand, for every $t \in K$ and $\lambda > 0$ we have

$$f(p_0(t) - \lambda w(t)) \le f(p_0(t)) + \lambda\varepsilon. \tag{3.137}$$

Hence

$$n[\psi(p_0-\lambda w)-\psi(p_0)]\le n\Big[f\big(p_0(t_n)-\frac{1}{n}w(t_n)\big)+\frac{\varepsilon}{n}-f\big(p_0(t_n)\big)\Big]. \tag{3.138}$$

Therefore, by (3.135), it follows that

$$\begin{aligned}-\frac{3\varepsilon}{2}&\le n\Big[\psi\big(p_0(t_n)-\frac{1}{n}w(t_n)\big)-f\big(p_0(t_n)\big)\Big]\\&\le n\Big[\psi\big(p_0(t_n)-\frac{1}{n}w(t_0)\big)-f\big(p_0(t_n)\big)\Big]\\&\quad+n\Big[f\big(p_0(t_n)-\frac{1}{n}w(t_n)\big)-f\big(p_0(t_n)-\frac{1}{n}w(t_0)\big)\Big].\end{aligned} \tag{3.139}$$

Since f is locally Lipschitz and $t_n\to t_0$, we deduce that

$$\limsup_{n\to\infty} n\Big[f\big(p_0(t_n)-\frac{1}{n}w(t_n)\big)-f\big(p_0(t_n)-\frac{1}{n}w(t_0)\big)\Big]=0. \tag{3.140}$$

Therefore,

$$-\frac{3\varepsilon}{2}\le\limsup_{n\to\infty} n\Big[f\big(p_0(t_0)+z_n-\frac{1}{n}w(t_0)\big)-f\big(p_0(t_0)+z_n\big)\Big], \tag{3.141}$$

where $z_n=p_0(t_n)-p_0(t_0)$. Hence

$$-\frac{3\varepsilon}{2}\le f^0\big(p_0(t_0),-w(t_0)\big)\le-\gamma+\varepsilon. \tag{3.142}$$

So

$$\gamma=\inf\{\|x^*\|;\ x^*\in\partial f(p_0(t)),t\in B(p_0)\}\le\frac{5\varepsilon}{2}. \tag{3.143}$$

Now, by the lower semicontinuity of λ, we find $t\in B(p_0)$ such that

$$\lambda(p_0(t))=\inf_{x^*\in\partial f(p_0(t))}\|x^*\|\le\frac{5\varepsilon}{2}, \tag{3.144}$$

which concludes our proof. □

Corollary 3.23. *Assume hypotheses of Theorem 3.22 are fulfilled and, moreover, f satisfies the Palais-Smale condition* $(\mathrm{PS})_c$. *Then c is a critical value of f.*

Remark 3.24. If

$$\inf_{x\in X_1} f(x+z)=\max_{x\in K^*} f(x), \tag{3.145}$$

then the conclusion of Corollary 3.17 remains valid, with an argument based on Theorem 1.20.

Corollary 3.25 (Ghoussoub-Preiss theorem). *Let $f : X \to \mathbb{R}$ be a locally Lipschitz Gâteaux-differentiable functional such that $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Let a and b be in X and define*

$$c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} f(p(t)), \tag{3.146}$$

where $\mathcal{P}$ is the set of continuous paths $p : [0,1] \to X$ such that $p(0) = a$ and $p(1) = b$. Let F be a closed subset of X which does not contain a and b and $f(x) \geq c$, for all $x \in F$. Suppose, in addition, that, for every $p \in \mathcal{P}$,

$$p([0,1]) \cap \mathcal{P} \neq \emptyset. \tag{3.147}$$

Then there exists a sequence (x_n) in X so that
(i) $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0$,
(ii) $\lim_{n\to\infty} f(x_n) = c$,
(iii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.
Moreover, if f satisfies $(\mathrm{PS})_c$*, then there exists $x \in F$ such that $f(x) = c$ and $f'(x) = 0$.*

Proof. With the same arguments as in the proof of Corollary 3.16, we deduce that the functional f is locally Lipschitz and

$$\partial f(x) = \{f'(x)\}. \tag{3.148}$$

Applying Theorem 3.22 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = a$, $p^*(1) = b$, our conclusion follows.

The last part of the theorem follows from Corollary 3.23. □

3.4. Applications of the mountain pass theorem

3.4.1. The subcritical Lane-Emden equation

Let $1 < p < (N+2)/(N-2)$, if $N \geq 3$, and $1 < p < +\infty$, provided that $N = 1, 2$. Consider the nonlinear elliptic boundary value problem

$$\begin{aligned} -\Delta u = u^p \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.149}$$

This equation is called the *Lane-Emden equation* and the existence results are related not only to the values of p, but also to the geometry of Ω. For instance, problem (3.149) has no solution if $p \geq (N+2)/(N-2)$ and if Ω is a *star-shaped domain* with respect to a certain point (the proof uses the *Pohozaev identity*, which is obtained after multiplication in (3.149) with $x \cdot \nabla u$ and integration by parts). If Ω is *not* star-shaped, Kazdan and Warner proved in [118] that problem (3.149) has a solution for *any* $p > 1$, where Ω is an *annulus* in $\mathbb{R}^N$.

If $p = (N+2)/(N-2)$ then the energy functional associated to problem (3.149) does not have the Palais-Smale property. The case $p \geq (N+2)/(N-2)$ is difficult; for instance, there is no solution even in the simplest case where $\Omega = B(0,1)$. If $p = 1$ then the existence of a solution depends on the geometry of the domain: if 1 is not an eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$, then there is no solution to our problem (3.149). If $0 < p < 1$, then there exists a unique solution (since the mapping $u \mapsto f(u)/u = u^{p-1}$ is decreasing) and, moreover, this solution is stable. The arguments may be done in this case by using the method of sub- and supersolutions.

Under the same assumptions on the subcritical exponent p, similar arguments show that the boundary value problem

$$\begin{gathered} -\Delta u - \lambda u = u^p \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.150}$$

has a solution for any $\lambda < \lambda_1$, where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. The proof of this result relies on the fact that the operator $(-\Delta - \lambda I)$ is coercive if $\lambda < \lambda_1$. Moreover, by multiplication with φ_1 and integration on Ω we deduce that there is no solution if $\lambda \geq \lambda_1$, where φ_1 stands for the first eigenfunction of the Laplace operator.

Our aim is to prove in what follows the following result.

Theorem 3.26. *There exists a solution of problem* (3.149)*, which is not necessarily unique. Furthermore, this solution is unstable.*

Proof. We first establish the instability of an arbitrary solution u. So, in order to argue that $\lambda_1(-\Delta - pu^{p-1}) < 0$, let $\varphi > 0$ be an eigenfunction corresponding to λ_1. We have

$$-\Delta\varphi - pu^{p-1}\varphi = \lambda_1\varphi \quad \text{in } \Omega. \tag{3.151}$$

Integrating by parts this equality, we find

$$(1-p)\int_\Omega u^p\varphi = \lambda_1\int_\Omega \varphi u, \tag{3.152}$$

which implies $\lambda_1 < 0$, since $u > 0$ in Ω and $p > 1$.

Next, we prove the existence of a solution by using two different methods.

(1) A variational proof. Let

$$m = \inf\left\{\int_\Omega |\nabla v|^2; v \in H_0^1(\Omega),\ \|v\|_{L^{p+1}} = 1\right\}. \tag{3.153}$$

First step (m is achieved). Let $(u_n) \subset H_0^1(\Omega)$ be a minimizing sequence. Since $p < (N+2)/(N-2)$, then $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$. It follows that

$$\int_\Omega |\nabla u_n|^2 \longrightarrow m \quad \text{as } n \longrightarrow \infty \tag{3.154}$$

and, for all positive integer n,

$$\|u_n\|_{L^{p+1}} = 1. \tag{3.155}$$

So, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u, \quad \text{weakly in } H_0^1(\Omega), \\ u_n &\longrightarrow u, \quad \text{strongly in } L^{p+1}(\Omega). \end{aligned} \tag{3.156}$$

By the lower semicontinuity of the functional $\|\cdot\|_{L^2}$, we find that

$$\int_\Omega |\nabla u|^2 \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 = m \tag{3.157}$$

which implies $\int_\Omega |\nabla u|^2 = m$. Since $\|u\|_{L^{p+1}} = 1$, it follows that m is achieved by u.

We remark that we have even $u_n \to u$, strongly in $H_0^1(\Omega)$. This follows by the weak convergence of (u_n) in $H_0^1(\Omega)$ and by $\|u_n\|_{H_0^1} \to \|u\|_{H_0^1}$.

Second step ($u \ge 0$, a.e. in Ω). We may assume that $u \ge 0$, a.e. in Ω. Indeed, if not, we may replace u by $|u|$. This is possible since $|u| \in H_0^1(\Omega)$ and so, by Stampacchia's theorem,

$$\nabla |u| = (\operatorname{sign} u)\nabla u \quad \text{if } u \ne 0. \tag{3.158}$$

Moreover, on the level set $[u = 0]$ we have $\nabla u = 0$, so

$$\big|\nabla |u|\big| = |\nabla u|, \quad \text{a.e. in } \Omega. \tag{3.159}$$

Third step (u verifies $-\Delta u = u^p$ in the weak sense). We have to prove that for every $w \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla w = m \int_\Omega u^p w. \tag{3.160}$$

Put $v = u + \varepsilon w$ in the definition of m. It follows that

$$\begin{aligned} \int_\Omega |\nabla v|^2 &= \int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \nabla w + \varepsilon^2 \int_\Omega |\nabla w|^2, \\ \int_\Omega |u + \varepsilon w|^{p+1} &= \int_\Omega |u|^{p+1} + \varepsilon(p+1)\int_\Omega u^p w + o(\varepsilon) \\ &= 1 + \varepsilon(p+1)\int_\Omega u^p w + o(\varepsilon). \end{aligned} \tag{3.161}$$

Therefore,

$$\|v\|_{L^{p+1}}^2 = \left(1 + \varepsilon(p+1)\int_\Omega u^p w + o(\varepsilon)\right)^{2/(p+1)} = 1 + 2\varepsilon \int_\Omega u^p w + o(\varepsilon). \tag{3.162}$$

Hence

$$\begin{aligned} m &= \int_\Omega |u|^2 \\ &\le \frac{m + 2\varepsilon \int_\Omega \nabla u \nabla w + o(\varepsilon)}{1 + 2\varepsilon \int_\Omega u^p w + o(\varepsilon)} \\ &= m + 2\varepsilon\left(\int_\Omega \nabla u \nabla w - m \int_\Omega u^p w\right) + o(\varepsilon), \end{aligned} \tag{3.163}$$

which implies

$$\int_\Omega \nabla u \nabla w = m \int_\Omega u^p w, \quad \text{for every } w \in H_0^1(\Omega). \tag{3.164}$$

Consequently, the function $u_1 = m^\alpha u$, where $\alpha = (p-1)^{-1}$, is a weak solution of problem (3.149). Thus, $u = u_1 m^{-\alpha}$ is a weak solution of problem (3.149).

Fourth step (regularity of u). We know until now that $u \in H_0^1(\Omega) \subset L^{2^\star}(\Omega)$. In a general framework, assuming that $u \in L^q$, it follows that $u^p \in L^{q/p}$, that is, by Schauder regularity and Sobolev embeddings, $u \in W^{2,q/p} \subset L^s$, where $1/s = p/q - 2/N$. So, assuming that $q_1 > (p-1)(N/2)$, we have $u \in L^{q_2}$, where $1/q_2 = p/q_1 - 2/N$. In particular, $q_2 > q_1$. Let (q_n) be the increasing sequence we may construct in this manner and set $q_\infty = \lim_{n\to\infty} q_n$. Assuming, by contradiction, that $q_n < Np/2$, we obtain, passing at the limit as $n \to \infty$, that $q_\infty = N(p-1)/2 < q_1$, contradiction. This shows that there exists $r > N/2$ such that $u \in L^r(\Omega)$ which implies $u \in W^{2,r}(\Omega) \subset L^\infty(\Omega)$. Therefore, $u \in W^{2,r}(\Omega) \subset C^k(\overline{\Omega})$, where k denotes the integer part of $2 - N/r$. Now, by Hölder continuity, $u \in C^2(\overline{\Omega})$.

(2) Second proof. The below arguments rely upon the mountain pass theorem. Set

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega (u^+)^{p+1}, \quad u \in H_0^1(\Omega). \tag{3.165}$$

Standard arguments show that F is a C^1 functional and u is a critical point of F if and only if u is a solution to problem (3.149). We observe that $F'(u) = -\Delta u - (u^+)^p \in H^{-1}(\Omega)$. So, if u is a critical point of F, then $-\Delta u = (u^+)^p \geq 0$ in Ω and hence, by the maximum principle, $u \geq 0$ in Ω.

We verify the hypotheses of the mountain pass theorem. Obviously, $F(0) = 0$. On the other hand,

$$\int_\Omega (u^+)^{p+1} \leq \int_\Omega |u|^{p+1} = \|u\|_{L^{p+1}}^{p+1} \leq C\|u\|_{H_0^1}^{p+1}. \tag{3.166}$$

Therefore,

$$F(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{C}{p+1}\|u\|_{H_0^1}^{p+1} \geq \rho > 0, \tag{3.167}$$

provided that $\|u\|_{H_0^1} = R$ is small enough.

Let us now prove the existence of some v_0 such that $\|v_0\| > R$ and $F(v_0) \leq 0$. For this aim, choose an arbitrary $w_0 \geq 0$, $w_0 \not\equiv 0$. We have

$$F(tw_0) = \frac{t^2}{2}\int_\Omega |\nabla w_0|^2 - \frac{t^{p+1}}{p+1}\int_\Omega (w_0^+)^{p+1} \leq 0, \tag{3.168}$$

for $t > 0$ large enough. □

3.4.2. A bifurcation problem

Let us consider a C^1 convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and $f'(0) > 0$. We assume that f has a subcritical growth, that is, there exists $1 < p < (N+2)/(N-2)$ such that for all $u \in \mathbb{R}$,

$$|f(u)| \leq C(1 + |u|^p). \tag{3.169}$$

We also suppose that there exist $\mu > 2$ and $A > 0$ such that

$$\mu \int_0^u f(t)dt \leq uf(u), \quad \text{for every } u \geq A. \tag{3.170}$$

A standard example of function satisfying these conditions is $f(u) = (1+u)^p$.

Consider the bifurcation problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.171}$$

We already know (by the implicit function theorem) that there exists $\lambda^\star > 0$ such that for every $\lambda < \lambda^\star$, there exists a minimal and stable solution $\underline{u}$ to problem (3.171).

Theorem 3.27. *Under the above hypotheses on f, for every $\lambda \in (0, \lambda^\star)$, there exists a second solution $u \geq \underline{u}$ and, furthermore, u is unstable.*

Proof. We find a solution u of the form $u = \underline{u} + v$ with $v \geq 0$. It follows that v satisfies

$$\begin{aligned} -\Delta v &= \lambda\big(f(\underline{u} + v) - f(\underline{u})\big) \quad \text{in } \Omega, \\ v &> 0 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.172}$$

Hence v fulfills an equation of the form

$$-\Delta v + a(x)v = g(x, v) \quad \text{in } \Omega, \tag{3.173}$$

where $a(x) = -\lambda f'(\underline{u})$ and

$$g(x, v) = \lambda\big(f(\underline{u}(x) + v) - f(\underline{u}(x))\big) - \lambda f'(\underline{u}(x))v. \tag{3.174}$$

We verify easily the following properties:

(i) $g(x, 0) = g_v(x, 0) = 0$;

(ii) $|g(x, v)| \leq C(1 + |v|^p)$;

(iii) $\mu \int_0^v g(x, t)dt \leq vg(x, v)$, for every $v \geq A$ large enough;

(iv) the operator $-\Delta - \lambda f'(\underline{u})$ is coercive, since $\lambda_1(-\Delta - \lambda f'(\underline{u})) > 0$, for every $\lambda < \lambda^\star$.

So, by the mountain pass theorem, problem (3.171) has a solution which is, a fortiori, unstable. □

3.5. Critical points and coerciveness of locally Lipschitz functionals with the strong Palais-Smale property

The Palais-Smale property for C^1 functionals appears as the most natural compactness condition. In order to obtain corresponding results for nondifferentiable functionals, the Palais-Smale condition introduced in Definition 3.5 is not always an efficient tool, because of the nonlinearity of the Clarke subdifferential. For this aim, we define a stronger Palais-Smale type condition, which is very useful in applications. In many cases, our compactness condition is a local one, similar to $(\mathrm{PS})_c$ introduced in Definition 3.5. The most efficient tool in our reasonings will be, as in the preceding paragraph, the Ekeland variational principle. As we will remark, the new Palais-Smale condition is in closed link with coerciveness properties of locally Lipschitz functionals.

As above, X denotes a real Banach space.

Definition 3.28. The locally Lipschitz functional $f : X \to \mathbb{R}$ is said to satisfy the strong Palais-Smale condition at the point c (notation: $(\mathrm{s}-\mathrm{PS})_c$) provided that for every sequence (x_n) in X satisfying

$$\lim_{n\to\infty} f(x_n) = c, \tag{3.175}$$

$$f^0(x_n, v) \geq -\frac{1}{n}\|v\|, \quad \text{for every } v \in X, \tag{3.176}$$

contains a convergent subsequence.

If this property holds for any real number c, we say that f satisfies the strong Palais-Smale condition $(\mathrm{s}-\mathrm{PS})$).

Remark 3.29. It follows from the continuity of f and the upper semicontinuity of $f^0(\cdot,\cdot)$ that if f satisfies the condition $(\mathrm{s}-\mathrm{PS})_c$ and there exist sequences (x_n) and (ε_n) such that the conditions (3.175) and (3.176) are fulfilled, then c is a critical value of f. Indeed, up to a subsequence, $x_n \to x$. It follows that $f(x) = c$ and, for every $v \in X$,

$$f^0(x, v) \geq \limsup_{n\to\infty} f^0(x_n, v) \geq 0, \tag{3.177}$$

that is, $0 \in \partial f(x)$.

Definition 3.30. A mapping $f : X \to \mathbb{R}$ is said to be coercive provided that

$$\lim_{\|x\|\to\infty} f(x) = +\infty. \tag{3.178}$$

For each $a \in \mathbb{R}$ and $f : X \to \mathbb{R}$, we denote

$$\begin{aligned} [f = a] &= \{x \in X; f(x) = a\}, \\ [f \leq a] &= \{x \in X; f(x) \leq a\}, \\ [f \geq a] &= \{x \in X; f(x) \geq a\}. \end{aligned} \tag{3.179}$$

Proposition 3.31. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional. If $a = \inf_X f$ and f satisfies the condition $(s - \mathrm{PS})_a$, then there exists $\alpha > 0$ such that the set $[f \leq a + \alpha]$ is bounded.*

Proof. We assume, by contradiction, that for every $\alpha > 0$, the set $[f \leq a+\alpha]$ is unbounded. So, there exists a sequence (z_n) in X such that for every $n \geq 1$,

$$a \leq f(z_n) \leq a + \frac{1}{n^2}, \qquad ||z_n|| \geq n. \tag{3.180}$$

Using Ekeland's variational principle, for every $n \geq 1$ there is some $x_n \in X$ such that for any $x \in X$,

$$\begin{gathered} a \leq f(x_n) \leq f(z_n), \\ f(x) - f(x_n) + \frac{1}{n}||x - x_n|| \geq 0, \\ ||x_n - z_n|| \leq \frac{1}{n}. \end{gathered} \tag{3.181}$$

Therefore,

$$\begin{gathered} ||x_n|| \geq n - \frac{1}{n} \longrightarrow \infty, \\ f(x_n) \longrightarrow a, \end{gathered} \tag{3.182}$$

and, for each $v \in X$,

$$f^0(x_n, v) \geq -\frac{1}{n}\|v\|. \tag{3.183}$$

Now, by the Palais-Smale condition $(\mathrm{s} - \mathrm{PS})_a$ we deduce that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. □

The following property is an immediate consequence of the above result.

Corollary 3.32. *If f is a locally Lipschitz bounded from below functional satisfying the strong Palais-Smale condition, then f is coercive.*

This result was proved by Li [134] for C^1 functionals. He used in his proof the deformation lemma. Corollary 3.32 also follows from the following property.

Proposition 3.33. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying*

$$a = \liminf_{\|x\| \to \infty} f(x) < +\infty. \tag{3.184}$$

Then there exists a sequence (x_n) in X such that

$$||x_n|| \longrightarrow \infty, \qquad f(x_n) \longrightarrow a \tag{3.185}$$

and, for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{1}{n}\|v\|. \tag{3.186}$$

Proof. For every $r > 0$ we define

$$m(r) = \inf_{\|x\| \geq r} f(x). \tag{3.187}$$

Obviously, the mapping m is nondecreasing and $\lim_{r\to\infty} m(r) = a$. For any integer $n \geq 1$, there exists $r_n > 0$ such that for every $r \geq r_n$,

$$m(r) \geq a - \frac{1}{n^2}. \tag{3.188}$$

Remark that we can choose r_n so that $r_n \geq n/2 + 1/n$. Choose $z_n \in X$ such that $\|z_n\| \geq 2r_n$ and

$$f(z_n) \leq m(r_{2n}) + \frac{1}{n^2} \leq a + \frac{1}{n^2}. \tag{3.189}$$

Applying Ekeland's variational principle to the functional f restricted to the set $\{x \in X;\ \|x\| \geq r_n\}$ and for $\varepsilon = 1/n$, $z = z_n$, we find $x_n \in X$ such that $\|x_n\| \geq r_n$ and, for every $x \in X$ with $\|x\| \geq r_n$,

$$f(x) \geq f(x_n) - \frac{1}{n}||x - x_n||, \tag{3.190}$$

$$a - \frac{1}{n^2} \leq m(r_n) \leq f(x_n) \leq f(z_n) - \frac{1}{n}||z_n - x_n||. \tag{3.191}$$

It follows from (3.189) and (3.191) that $\|x_n - z_n\| \leq 2/n$, which implies

$$||x_n|| \geq 2r_n - \frac{2}{n} \longrightarrow +\infty. \tag{3.192}$$

On the other hand, $f(x_n) \to a$. For every $v \in X$ and $\lambda > 0$, putting $x = x_n + \lambda v$ in (3.190), we find

$$f^0(x_n, v) \geq \limsup_{\lambda \searrow 0} \frac{f(x_n + \lambda v) - f(x_n)}{\lambda} \geq -\frac{1}{n}\|v\|, \tag{3.193}$$

which concludes our proof. □

Proposition 3.34. *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz bounded from below functional. Assume there exists* $c \in \mathbb{R}$ *such that* f *satisfies* $(s - \mathrm{PS})_c$ *and, for every* $a < c$, *the set* $[f \leq a]$ *is bounded.*

Then there exists $\alpha > 0$ *such that the set* $[f \leq c + \alpha]$ *is bounded.*

Proof. Arguing by contradiction, we assume that the set $[f \leq c + \alpha]$ is unbounded, for every $\alpha > 0$. It follows by our hypothesis that, for every $n \geq 1$, there exists $r_n \geq n$ such that

$$\left[f \leq c - \frac{1}{n^2}\right] \subset B(0, r_n). \tag{3.194}$$

Set

$$c_n = \inf_{X \setminus B(0,r_n)} f \geq c - \frac{1}{n^2}. \tag{3.195}$$

Since the set $[f \leq c + 1/n^2]$ is unbounded, we obtain the existence of a sequence (z_n) in X such that

$$\begin{gathered} \|z_n\| \geq r_n + 1 + \frac{1}{n}, \\ f(z_n) \leq c + \frac{1}{n^2}. \end{gathered} \tag{3.196}$$

So, $z_n \in X \setminus B(0, r_n)$ and

$$f(z_n) \leq c_n + \frac{2}{n^2}. \tag{3.197}$$

Applying Ekeland's variational principle to the functional f restricted to $X \setminus B(0, r_n)$, we find $x_n \in X \setminus B(0, r_n)$ such that for every $x \in X$ with $\|x\| \geq r_n$, we have

$$\begin{gathered} c_n \leq f(x_n) \leq f(z_n), \\ f(x) \geq f(x_n) - \frac{2}{n}\|x - x_n\|, \\ \|x_n - z_n\| \leq \frac{1}{n}. \end{gathered} \tag{3.198}$$

Hence

$$\begin{gathered} \|x_n\| \geq \|z_n\| - \|x_n - z_n\| \geq r_n + 1 \longrightarrow +\infty, \\ f(x_n) \longrightarrow c \end{gathered} \tag{3.199}$$

and, for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{2}{n}\|v\|. \tag{3.200}$$

Now, by $(\mathrm{s} - \mathrm{PS})_c$, we deduce that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. □

Proposition 3.35. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional. Assume that f is not coercive and denote*

$$c = \sup \{a \in \mathbb{R};\ [f \leq a] \text{ is bounded}\}. \tag{3.201}$$

Then f does not satisfy the condition $(s - \mathrm{PS})_c$.

Proof. Set

$$A = \{a \in \mathbb{R};\ [f \leq a] \text{ is bounded}\}. \tag{3.202}$$

It follows from the lower boundedness of f that the set A is nonempty. Since f is not coercive, it follows that

$$c = \sup A < +\infty. \tag{3.203}$$

Assume, by contradiction, that f satisfies $(\mathrm{s}-\mathrm{PS})_c$. Then, by Proposition 3.34, there exists $\alpha > 0$ such that the set $[f \le a + \alpha]$ is bounded, which contradicts the maximality of c. □

Remark 3.36. The real number c defined in Proposition 3.34 may also be characterized by

$$c = \inf\{b \in \mathbb{R};\ [f \le b] \text{ is unbounded}\}. \tag{3.204}$$

Proposition 3.37. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying* $(s-\mathrm{PS})$. *Assume there exists $a \in \mathbb{R}$ such that the set $[f \le a]$ is bounded.*

Then the functional f is coercive.

Proof. Without loss of generality, we may assume that $a = 0$. It follows now from our hypothesis that there exists an integer n_0 such that $f(x) > 0$, for any $x \in X$ with $\|x\| \ge n_0$. We assume, by contradiction, that

$$0 \le c = \liminf_{\|x\| \to \infty} f(x) < +\infty. \tag{3.205}$$

Applying Proposition 3.33, we find a sequence (x_n) in X such that $\|x_n\| \to \infty$, $f(x_n) \to c$ and, for every $v \in X$,

$$f^0(x_n, v) \ge -\frac{1}{n}\|v\|. \tag{3.206}$$

Using now the condition $(\mathrm{s}-\mathrm{PS})$, we deduce that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. So, the functional f is coercive. □

Corollary 3.38. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional which satisfies* $(s-\mathrm{PS})$.

Then every minimizing sequence of f contains a convergent subsequence.

Proof. Let (x_n) be a minimizing sequence of f. Passing eventually at a subsequence, we have

$$f(x_n) \le \inf_X f + \frac{1}{n^2}. \tag{3.207}$$

By Ekeland's variational principle, there exists $z_n \in X$ such that for every $x \in X$,

$$\begin{aligned} f(x) &\ge f(z_n) - \frac{1}{n}||x - z_n||, \\ f(z_n) &\le f(x_n) - \frac{1}{n}||x_n - z_n||. \end{aligned} \tag{3.208}$$

With an argument similar to that used in the proof of Proposition 3.33, we find

$$||x_n - z_n|| \leq \frac{2}{n}, \tag{3.209}$$

$$f(z_n) \leq \inf_X f + \frac{1}{n^2}, \tag{3.210}$$

and, for every $v \in X$,

$$f^0(z_n, v) \geq -\frac{1}{n}\|v\|. \tag{3.211}$$

Using now the condition (s − PS), we deduce that the sequence (z_n) is relatively compact. By (3.209) it follows that the corresponding subsequence of (x_n) is convergent, too. □

Define the map

$$M : [0, +\infty) \longrightarrow \mathbb{R}, \quad M(r) = \inf_{\|x\|=r} f(x). \tag{3.212}$$

We prove in what follows some elementary properties of this functional.

Proposition 3.39. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional which satisfies the condition* (s − PS). *Assume there exists $R > 0$ such that all the critical points of f are in the closed ball of radius R.*

Then the functional M is increasing and continuous at the right on the set $(R, +\infty)$.

For the proof of this result, we use an auxiliary property. First we introduce a weaker variant of the condition (s − PS) for functionals defined on a circular crown.

Definition 3.40. Let $0 < a < b$ and let f be a locally Lipschitz map defined on

$$A = \{x \in X;\ a \leq \|x\| \leq b\}. \tag{3.213}$$

We say that f satisfies the Palais-Smale-type condition $(\mathrm{PS})_A$ provided that every sequence (x_n) satisfying

$$\begin{gathered} a + \delta \leq ||x_n|| \leq b - \delta, \quad \text{for some } \delta > 0, \\ \sup_n |f(x_n)| < +\infty, \\ f^0(x_n, v) \geq -\frac{1}{n}\|v\|, \quad \text{for some } v \in X, \end{gathered} \tag{3.214}$$

contains a convergent subsequence.

Lemma 3.41. *Let A be as in Definition 3.40 and let f be a locally Lipschitz bounded from below functional defined on A. Assume that f satisfies* $(\mathrm{PS})_A$ *and f does not have critical points which are interior points of A.*

Then, for every $a < r_1 < r < r_2 < b$,

$$M(r) > \min\{M(r_1), M(r_2)\}. \tag{3.215}$$

Proof. Without loss of generality, let us assume that f takes only positive values. Arguing by contradiction, let $r_1 < r < r_2$ be such that inequality (3.215) is not fulfilled. There exists a sequence (x_n) such that $\|x_n\| = r$ and

$$f(x_n) < M(r) + \frac{1}{n^2}. \tag{3.216}$$

Applying now Ekeland's variational principle to f restricted to the set

$$B = \{x \in X;\ r_1 \le \|x\| \le r_2\}, \tag{3.217}$$

we find $z_n \in B$ such that, for every $x \in B$,

$$\begin{aligned} f(x) &\ge f(z_n) - \frac{1}{n}\|x - z_n\|, \\ f(z_n) &\le f(x_n) - \frac{1}{n}\|x_n - z_n\|. \end{aligned} \tag{3.218}$$

Moreover, $r_1 < \|z_n\| < r_2$, for n large enough. Indeed, if it would exist $n \ge 1$ such that $\|z_n\| = r_1$, then

$$\begin{aligned} M(r_1) &\le f(x_n) - \frac{1}{n}\|x_n - z_n\| \le M(r) + \frac{1}{n^2} - \frac{1}{n}\|x_n - z_n\| \\ &\le M(r) + \frac{1}{n^2} - \frac{1}{n}(r - r_1) \le M(r_1) + \frac{1}{n^2} - \frac{1}{n}(r - r_1). \end{aligned} \tag{3.219}$$

It follows that $r - r_1 \le 1/n$, which is not possible if n is sufficiently large. Therefore,

$$\sup_n |f(z_n)| = \sup_n f(z_n) \le M(r) \tag{3.220}$$

and, for every $v \in X$,

$$f^0(z_n, v) \ge -\frac{1}{n}\|v\|. \tag{3.221}$$

Using now $(\mathrm{PS})_A$, the sequence (z_n) contains a subsequence which converges to a critical point of f belonging to B. This contradicts one of the hypotheses imposed to f. □

Proof of Proposition 3.39. If M is not increasing, there exists $r_1 < r_2$ such that $M(r_2) \le M(r_1)$. On the other hand, by Corollary 3.32 we have

$$\lim_{r \to \infty} M(r) = +\infty. \tag{3.222}$$

Choosing now $r > r_2$ so that $M(r) \ge M(r_1)$, we find that $r_1 < r_2 < r$ and

$$M(r_2) \le M(r_1) = \min\{M(r_1), M(r)\}, \tag{3.223}$$

which contradicts Lemma 3.41. So, M is an increasing map.

The continuity at the right of M follows from its upper semicontinuity. □

There exists a local variant of Proposition 3.39 for locally Lipschitz functionals defined on the set $\{x \in X;\ \|x\| \le R_0\}$, for some $R_0 > 0$.

Assume f satisfies the condition (s − PS) in the following sense: every sequence (x_n) with the properties

$$\begin{gathered} \|x_n\| \le R < R_0, \\ \sup_n |f(x_n)| < +\infty, \\ f^0(x_n, v) \ge -\frac{1}{n}\|v\|, \quad \text{for every } v \in X,\ n \ge 1, \end{gathered} \tag{3.224}$$

is relatively compact.

Proposition 3.42. *Let f be a locally Lipschitz functional defined on $\|x\| \le R_0$ and satisfying the condition $(s - \mathrm{PS})$. Assume $f(0) = 0$, $f(x) > 0$ provided $0 < \|x\| < R_0$, and assume f does not have critical point in the set $\{x \in X;\ 0 < \|x\| < R_0\}$.*

Then there exists $0 < r_0 \le R_0$ such that M is increasing on $[0, r_0)$ and decreasing on $[r_0, R_0)$.

Proof. Let (R_n) be an increasing sequence of positive numbers which converges to R_0. By the upper semicontinuity of M restricted to $[0, R_n]$ it follows that there exists $r_n \in (0, R_n]$ such that M achieves its maximum in r_n. Let $r_0 \in (0, R_0]$ be the limit of the increasing sequence (r_n). Our conclusion now follows from Lemma 3.41. □

3.6. Bibliographical notes

The mountain pass theorem was established by Ambrosetti and Rabinowitz in [14]. Their original proof relies on some deep deformation techniques developed in the papers of Palais (see, e.g., [174]). Brezis and Nirenberg provided in [33] a simpler proof which combines two major tools: Ekeland's variational principle and the pseudo-gradient lemma. Ekeland's variational principle is the nonlinear version of the Bishop-Phelps theorem and it may be also viewed as a generalization of Fermat's theorem. The case of mountains of zero altitude (Corollary 10) is due to Pucci and Serrin [179, 180]. Another important generalization is due to Ghoussoub and Preiss (see Corollary 3.25 in this section).

In the framework of locally Lipschitz functionals, Theorems 3.11, 3.19, and 3.22 are due to Rădulescu [191]. The relationship between critical points and coerciveness is also due to Rădulescu [191].

4 Nonsmooth Ljusternik-Schnirelmann theory

We believe that the human mind is a "meteor" in the same way as the rainbow—a natural phenomenon; and that Hilbert realizing the "spectral decomposition" of linear operators, Perrin analyzing the blue color of the sky, Monet, Debussy, and Proust recreating, for our wonder, the scintillation of the light on the see, all worked for the same aim, which will also be that of the future: the knowledge of thewhole Universe.

Roger Godement

A great impact in the calculus of variations, in connection with differential equations, is the work of Ljusternik and Schnirelmann [140], which is a generalization of the min-max theory of Birkhoff [26]. In 1930, Ljusternik [139] proved that a C^1 even function on the sphere S^{n-1} has at least n pairs of critical points. This result is related to the antipodal theorem of Borsuk [28]. Ljusternik and Schnirelmann [140] introduced important min-max methods for obtaining multiple critical points of functions defined on manifolds.

In this chapter, we establish two Ljusternik-Schnirelmann-type theorems for nonsmooth functionals defined on real Banach spaces. The first result relies on the notion of critical point for a pairing of operators, which was introduced by Fucik et.al. [82]. The second theorem is related to locally Lipschitz functionals which are periodic with respect to a discrete subgroup. Our framework in this case uses the notion of Clarke subdifferential associated to a locally Lipschitz functional.

4.1. Genus and Ljusternik-Schnirelmann category

One of the most interesting problems related to the extremum problems is how to find estimates of the eigenvalues and eigenfunctions of a given operator. In this field, the Ljusternik-Schnirelmann theory plays a very important role. The starting point of this theory is the eigenvalue problem

$$Ax = \lambda x, \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{4.1}$$

where $A \in M_n(\mathbb{R})$ is a symmetric matrix. This problem may be written, equivalently,

$$F'(x) = \lambda x, \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{4.2}$$

where

$$F(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i x_j, \tag{4.3}$$

provided that $A = (a_{ij})$ and $x = (x_1, \dots, x_n)$.

The eigenvalues of the operator A are, by Courant's min-max principle,

$$\lambda_k = \max_{M\in\mathcal{V}_k}\min_{x\in M}\frac{\langle Ax, x\rangle}{\|x\|^2} = 2\max_{A\in\mathcal{V}_k}\min_{x\in M} F(x), \tag{4.4}$$

for $1 \le k \le n$, where $\mathcal{V}_k$ denotes the set of all vector subspaces of $\mathbb{R}^n$ with the dimension k.

The first result in the Ljusternik-Schnirelmann theory was proved in 1930 and is the following.

Ljusternik-Schnirelmann theorem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 even functional. Then f' has at least $2n$ distinct eigenfunctions on the sphere S^{n-1}.

The ulterior achievements in mathematics showed the significance of this theorem. We only point out that the variational arguments play at this moment a very strong instrument in the study of potential operators. Hence it is not a coincidence the detail that the solutions of such a problem are found by analyzing the extrema of a suitable functional.

L. Ljusternik and L. Schnirelmann developed their theory using the notion of *Ljusternik-Schnirelmann category* of a set. A simpler notion is that of *genus*, which is due to Coffman [49], but equivalent to that introduced by M. A. Krasnoselski.

Let X be a real Banach space and denote by $\mathcal{F}$ the family of all closed and symmetric with respect to the origin subsets of $X \setminus \{0\}$.

Definition 4.1. A nonempty subset A of $\mathcal{F}$ has the genus k provided k is the least integer with the property that there exists a continuous odd mapping $h : A \to \mathbb{R}^k \setminus \{0\}$.

We will denote from now on by $\gamma(A)$ the genus of the set $A \in \mathcal{F}$.

By definition, $\gamma(\emptyset) = 0$ and $\gamma(A) = +\infty$, if $\gamma(A) \neq k$, for every integer k.

Lemma 4.2. *Let $D \subset \mathbb{R}^n$ be a bounded open and symmetric set which contains the origin. Let $f : \overline{D} \to \mathbb{R}^n$ be a continuous function which does not vanish on the boundary of D. Then $f(D)$ contains a neighborhood of the origin.*

Proof. By Borsuk's theorem, the Brouwer degree $d[f; D, 0]$ is an odd number, so it is different from 0. Now, by the existence theorem for the topological degree, it follows that $0 \in f(D)$. Next, the property of continuity of the topological degree implies the existence of some $\varepsilon > 0$ such that $a \in f(D)$, for all $a \in \mathbb{R}^n$ with $\|a\| < \varepsilon$. □

Lemma 4.3. *Let D be as in Lemma 4.2 and assume that $g : \partial D \to \mathbb{R}^n$ is a continuous odd function, such that the set $g(\partial D)$ is contained in a proper subspace of $\mathbb{R}^n$.*

Then there exists $z \in \partial D$ such that $g(z) = 0$.

Proof. We may suppose that $g(\partial D) \subset \mathbb{R}^{n-1}$. If g does not vanish on ∂D, then, by Tietze's theorem, there exists an extension h of g at the set $\overline{D}$. By Lemma 4.2, the set $h(\overline{D})$ contains a neighborhood of the origin in $\mathbb{R}^n$, which is not possible, because $h(\overline{D}) \subset \mathbb{R}^{n-1}$. Thus, there is some $z \in \partial D$ such that $g(z) = 0$. □

Lemma 4.4. *Let $A \in \mathcal{F}$ a set which is homeomorphic with S^{n-1} by an odd homeomorphism. Then $\gamma(A) = n$.*

Proof. Obviously, $\gamma(A) \leq n$. If $\gamma(A) = k < n$, then there exists $h : A \to \mathbb{R}^k \setminus \{0\}$ continuous and odd. Let $f : A \to S^{n-1}$ the homeomorphism given in the hypothesis. Then $h \circ f^{-1} : S^{n-1} \to \mathbb{R}^k \setminus \{0\}$ is continuous and odd, which contradicts Lemma 4.3. Therefore, $\gamma(A) = n$. □

The main properties of the notion of genus of a closed and symmetric set are listed in what follows.

Lemma 4.5. *Let $A, B \in \mathcal{F}$.*

(i) *If there exists $f : A \to B$ continuous and odd then $\gamma(A) \leq \gamma(B)$.*
(ii) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(iii) *If the sets A and B are homeomorphic, then $\gamma(A) = \gamma(B)$.*
(iv) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
(v) *If $\gamma(B) < +\infty$, then*

$$\gamma(A) - \gamma(B) \leq \gamma(\overline{A \setminus B}). \tag{4.5}$$

(vi) *If A is compact, then $\gamma(A) < +\infty$.*
(vii) *If A is compact, then there exists $\varepsilon > 0$ such that*

$$\gamma\big(V_\varepsilon(A)\big) = \gamma(A), \tag{4.6}$$

where

$$V_\varepsilon(A) = \{x \in X;\ \operatorname{dist}(x, A) \leq \varepsilon\}. \tag{4.7}$$

Proof. (i) If $\gamma(B) = n$, let $h : B \to \mathbb{R}^n \setminus \{0\}$ continuous and odd. Then the mapping $h \circ f : A \to \mathbb{R}^n \setminus \{0\}$ is also continuous and odd, that is, $\gamma(A) \leq n$.

If $\gamma(B) = +\infty$, the result is trivial.

(ii) We choose $f = \mathrm{Id}$ in the preceding proof.

(iii) It follows from (i), by interchanging the sets A and B.

(iv) Let $\gamma(A) = m$, $\gamma(B) = n$ and let $f : A \to \mathbb{R}^m \setminus \{0\}$, $g : B \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd. By Tietze's theorem let $F : X \to \mathbb{R}^m$ and $G : X \to \mathbb{R}^n$ be continuous extensions of f and g. Moreover, let us assume that F and G are odd. If not, we replace the function F with

$$x \longmapsto \frac{F(x) - F(-x)}{2}. \tag{4.8}$$

Let

$$h = (F, G) : A \cup B \longrightarrow \mathbb{R}^{m+n} \setminus \{0\}. \tag{4.9}$$

Clearly, h is continuous and odd, that is $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.

(v) Follows from (ii), (iv), and the fact that $A \subset (\overline{A \setminus B}) \cup B$.

(vi) If $x \neq 0$ and $r < \|x\|$, then $B_r(x) \cap B_r(-x) = \emptyset$. So,

$$\gamma(B_r(x) \cup B_r(-x)) = 1. \tag{4.10}$$

By compactness arguments, we can cover the set A with a finite number of open balls, that is $\gamma(A) < +\infty$.

(vii) Let $\gamma(A) = n$ and $f : A \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd. With the same arguments as in (iv), let $F : X \to \mathbb{R}^n$ be a continuous and odd extension of f.

Since f does not vanish on the compact set A, there is some $\varepsilon > 0$ such that F does not vanish in $V_\varepsilon(A)$. Thus $\gamma(V_\varepsilon(A)) \le n = \gamma(A)$.

The reversed inequality follows from (ii). □

We give in what follows the notion of *Ljusternik-Schnirelmann category* of a set. For further details and proof we refer to the monograph by Mawhin and Willem [144] and the pioneering paper by Palais [174].

A topological space X is said to be *contractible* provided that the identic map is homotopic with a constant map, that is, there exist $u \in X$ and a continuous function $F : [0,1] \times X \to X$ such that, for every $x \in X$,

$$F(0,x) = x, \qquad F(1,x) = u. \tag{4.11}$$

A subset M of X is said to be *contractible in* X is there exist $u \in X$ and a continuous function $F : [0,1] \times M \to X$ such that, for every $x \in M$,

$$F(0,x) = x, \qquad F(1,x) = u. \tag{4.12}$$

If A is a subset of X, define the *category of* A *in* X, denoted by $\text{Cat}_X(A)$, as follows:

(i) $\text{Cat}_X(A) = 0$, if $A = \emptyset$;

(ii) $\text{Cat}_X(A) = n$, if n is the smallest integer such that A may be covered with n closed sets which are contractible in X;

(iii) $\text{Cat}_X(A) = \infty$, if contrary.

Lemma 4.6. *Let A and B be subsets of X.*

(i) *If $A \subset B$, then* $\text{Cat}_X(A) \le \text{Cat}_X(B)$.

(ii) $\text{Cat}_X(A \cup B) \le \text{Cat}_X(A) + \text{Cat}_X(B)$.

(iii) *Let $h : [0,1] \times A \to X$ be a continuous mapping such that $h(0,x) = x$, for every $x \in A$.*

If A is closed and $B = h(1,A)$, then $\text{Cat}_X(A) \le \text{Cat}_X(B)$.

The proof is straightforward and follows from the definition of $\text{Cat}_X(A)$. We refer to the monograph by Mawhin and Willem [144] for complete details.

4.2. A finite dimensional version of the Ljusternik-Schnirelmann theorem

The first version of the celebrated Ljusternik-Schnirelmann theorem, published in [140], was generalized in several directions. We prove in what follows a finite dimensional variant, by using the notion of *genus* of a set. Other variants of the Ljusternik-Schnirelmann theorem may be found in Krasnoselski [126], Palais [174], Rabinowitz [186], Struwe [214].

Let $f, g \in C^1(\mathbb{R}^n, \mathbb{R})$ and let $a > 0$ be a fixed real number.

Definition 4.7. We say that the functional f has a critical point with respect to g and a if there exist $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$f'(x) = \lambda g'(x),\qquad g(x) = a. \tag{4.13}$$

In this case, x is said to be a critical point of f (with respect to the mapping g and the number a), while $f(x)$ is called a critical value of f.

We say that the real number c is a critical value of f if problem (2.11) admits a solution $x \in \mathbb{R}^n$ such that $f(x) = c$.

Lemma 4.8. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be an even map which is Fréchet differentiable and such that*

(1) $\operatorname{Ker} g = \{0\}$,
(2) $\langle g'(x), x\rangle > 0$, *for every* $x \neq 0$,
(3) $\lim_{\|x\|\to\infty} g(x) = \infty$.

Then the sets $[g = a]$ and S^{n-1} are homeomorphic.

Proof. Let

$$h : [g = a] \longrightarrow S^{n-1}, \quad h(x) = \frac{x}{\|x\|}. \tag{4.14}$$

Evidently, h is well defined and continuous. We prove in what follows that h is one-to-one and onto.

Let $y \in S^{n-1}$. Consider the mapping

$$f : [0, \infty) \longrightarrow \mathbb{R}, \quad f(t) = g(ty). \tag{4.15}$$

Then f is differentiable and, for every $t > 0$,

$$f'(t) = \langle g'(ty), y\rangle > 0. \tag{4.16}$$

Since $f(0) > 0$ and $\lim_{t\to\infty} f(t) = \infty$, it follows that there exists a unique $t_0 > 0$ such that $f(t_0) = a$, that is, $g(t_0 y) = a$. Thus, $t_0 y \in [g = a]$ and $h(t_0 y) = y$. Thus, h is surjective.

Let now $x, y \in [g = a]$ be such that $h(x) = h(y)$, that is,

$$\frac{x}{\|x\|} = \frac{y}{\|y\|}. \tag{4.17}$$

If $x \neq y$, then there is some $t_0 > 0$, $t \neq 1$ such that $y = t_0 x$.
Consider the mapping

$$\psi : [0, \infty) \longrightarrow \mathbb{R}, \quad \psi(t) = g(tx). \tag{4.18}$$

It follows that $\psi(1) = \psi(t_0)$. But, for every $t > 0$,

$$\psi'(t) = \langle g'(tx), x \rangle > 0, \tag{4.19}$$

which implies that the equality $\psi(1) = \psi(t)$ is not possible provided $t \neq 1$. We deduce that $x = y$, that is, h is one-to-one.

Condition (3) from our hypotheses implies the continuity of h^{-1}. Moreover, h^{-1} is odd, because g is even. Thus, h is the desired homeomorphism. □

For every $1 \leq k \leq n$ define the set

$$\mathcal{V}_k = \{A;\ A \subset [g = a],\ A \text{ compact, symmetric and } \gamma(A) \geq k\}. \tag{4.20}$$

Let F be a vector subspace of $\mathbb{R}^n$ and of dimension k. If $S_k = F \cap S^{n-1}$, it follows by Lemma 4.8 that the set $A = h^{-1}(S_k)$ lies in $\mathcal{V}_k$, that is, $\mathcal{V}_k \neq \emptyset$, for every $1 \leq k \leq n$. Moreover,

$$\mathcal{V}_n \subset \mathcal{V}_{n-1} \subset \mathcal{V}_1. \tag{4.21}$$

Theorem 4.9. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two even functionals of class C^1 and $a > 0$ fixed. Assume that g satisfies the following assumptions:*

(1) $\operatorname{Ker} g = \{0\}$,
(2) $\langle g'(x), x \rangle > 0$, *for every* $x \in \mathbb{R}^n \setminus \{0\}$,
(3) $\lim_{\|x\| \to \infty} g(x) = \infty$.

Then f admits at least $2n$ critical points with respect to the application g and the number a.

Proof. Observe first that the critical points appear in pairs, because of the evenness of the mappings f and g.

Step 1. Characterization of the critical values of f. Let, for every $1 \leq k \leq n$,

$$c_k = \sup_{A \in \mathcal{V}_k} \min_{x \in A} f(x). \tag{4.22}$$

We propose to show that c_k are critical values of f. This is not enough for concluding the proof, since it is possible that the numbers c_k are not distinct.

If c is a real number, let

$$A_c = \{x \in [g = a];\ f(x) \geq c\}. \tag{4.23}$$

We will prove that, for every $1 \leq k \leq n$,

$$c_k = \sup \{r \in \mathbb{R};\ \gamma(A_r) \geq k\}. \tag{4.24}$$

Set

$$x_k = \sup\{r \in \mathbb{R};\ \gamma(A_r) \geq k\}. \tag{4.25}$$

From $\gamma(A_r) \geq k$ it follows that $\inf\{f(x);\ x \in A_r\} \leq c_k$, that is, $x_k \leq c_k$.

If $x_k < c_k$, then there exists $A \in \mathcal{V}_k$ such that

$$c_k > \inf_{x\in A} f(x) = \alpha > x_k. \tag{4.26}$$

Thus, $A \subset A_\alpha$ and $k \leq \gamma(A) \leq \gamma(A_\alpha)$, which contradicts the definition of x_k. Consequently, $c_k = x_k$.

Using now the fact that, for every $\varepsilon > 0$, it follows that

$$\gamma(A_{c_k-\varepsilon}) \geq k. \tag{4.27}$$

Let K_c be the set of critical values of f corresponding to the critical value c. By the deformation lemma (see [186, Theorem A.4]), if V is a neighborhood of K_c, there exist $\varepsilon > 0$ and $\eta \in C([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$ such that, for every fixed $t \in [0,1]$, the mapping

$$x \longmapsto \eta(t,x) \tag{4.28}$$

is odd and

$$\eta(1, A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}. \tag{4.29}$$

Now, putting for every $x \in \mathbb{R}^n$,

$$s(x) = \eta(1,x), \tag{4.30}$$

we obtain

$$s(A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}. \tag{4.31}$$

In particular, if $K_c = \varnothing$, then

$$s(A_{c-\varepsilon}) \subset A_{c+\varepsilon}. \tag{4.32}$$

Step 2. For every $1 \leq k \leq n$, the number c_k is a critical value of f.

Indeed, if not, using the preceding result, there exists $\varepsilon > 0$ such that

$$s(A_{c_k-\varepsilon}) \subset A_{c_k+\varepsilon}. \tag{4.33}$$

From $\gamma(A_{c_k-\varepsilon}) \geq k$. By Lemma 4.5(ii), it follows that

$$\gamma(s(A_{c_k-\varepsilon})) \geq k. \tag{4.34}$$

The definition of c_k yields

$$c_k \geq \inf_{x\in s(A_{c_k-\varepsilon})} f(x). \tag{4.35}$$

By (4.31) and (4.35), it follows that $c_k \geq c_k + \varepsilon$, contradiction.

Step 3. A multiplicity argument. We study in what follows the case of multiple critical values. Let us assume that

$$c_{k+1} = \cdots = c_{k+p} = c, \quad p > 1. \tag{4.36}$$

In this case, we prove that $\gamma(K_c) \geq p$.

If, by contradiction, $\gamma(K_c) \leq p-1$, then, by Lemma 4.5(vii), there is some $\varepsilon > 0$ such that

$$\gamma\big(V_\varepsilon(K_c)\big) \leq p - 1. \tag{4.37}$$

Let $V = \operatorname{Int} V_\varepsilon(K_c)$. By (4.31), it follows that

$$s(A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}. \tag{4.38}$$

Observe that

$$B = \overline{A_{c-\varepsilon} - V_\varepsilon(K_c)} = A_{c-\varepsilon} \setminus \operatorname{Int} V_\varepsilon(K_c). \tag{4.39}$$

But $\gamma(A_{c-\varepsilon}) \geq k + p$. Using now Lemma 4.5(v), we have

$$\gamma(B) \geq \gamma(A_{c-\varepsilon}) - \gamma\big(V_\varepsilon(K_c)\big) \geq k+1. \tag{4.40}$$

By Lemma 4.5(i), it follows that

$$\gamma\big(s(B)\big) \geq k+1. \tag{4.41}$$

The definition of $c = c_{k+1}$ shows that

$$\inf_{x \in s(B)} f(x) \leq c. \tag{4.42}$$

The inclusion $s(B) \subset A_{c+\varepsilon}$ implies

$$\inf_{x \in s(B)} f(x) \geq c + \varepsilon, \tag{4.43}$$

which contradicts (4.42). □

4.3. Critical points of locally Lipschitz Z-periodic functionals

Let X be a Banach space and let Z be a discrete subgroup of it. Therefore,

$$\inf_{z \in Z \setminus \{0\}} \|z\| > 0. \tag{4.44}$$

Definition 4.10. A function $f : X \to \mathbb{R}$ is said to be Z-periodic provided that $f(x+z) = f(x)$, for every $x \in X$ and $z \in Z$.

If the locally Lipschitz functional $f : X \to \mathbb{R}$ is Z-periodic, then, for every $v \in X$, the mapping $x \mapsto f^0(x, v)$ is Z-periodic and ∂f is Z-periodic, that is, for every $x \in X$ and $z \in Z$,

$$\partial f(x+z) = \partial f(x). \tag{4.45}$$

Thus, the functional λ inherits the property of Z-periodicity.

If $\pi : X \to X/Z$ is the canonical surjection and x is a critical point of f, then the set $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is said to be a *critical orbit* of f. We also remark that X/Z becomes a complete metric space if it is endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z \in Z} \|x - y - z\|. \tag{4.46}$$

A locally Lipschitz functional which is Z-periodic $f : X \to \mathbb{R}$ satisfies the Palais-Smale $(\mathrm{PS})_Z$ condition provided that, for every sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, there exists a convergent subsequence of $(\pi(x_n))$. Equivalently, this means that, up to a subsequence, there exists $z_n \in Z$ such that the sequence $(x_n - z_n)$ is convergent. If c is a real number, then f satisfies the local condition of type Palais-Smale $(\mathrm{PS})_{Z,c}$ if, for every sequence (x_n) in X such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$, there exists a convergent subsequence of $(\pi(x_n))$.

Theorem 4.11. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional which is Z-periodic and satisfies the assumption* (3.34).

If f satisfies the condition $(\mathrm{PS})_{Z,c}$, then c is a critical value of f, corresponding to a critical point which is not in $\pi^{-1}(\pi(p^*(K^*)))$.

Proof. With the same arguments as in the proof of Theorem 3.11, we find a sequence (x_n) in X such that

$$\lim_{n\to\infty} f(x_n) = c, \qquad \lim_{n\to\infty} \lambda(x_n) = 0. \tag{4.47}$$

The Palais-Smale condition $(\mathrm{PS})_{Z,c}$ implies the existence of some x such that, up to a subsequence, $\pi(x_n) \longrightarrow \pi(x)$. Passing now to the equivalence class mod Z, we may assume that $x_n \to x$ in X. Moreover, x is a critical point of f, because

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(x_n) = c, \\ \lambda(x) &\le \liminf_{n\to\infty} \lambda(x_n) = 0. \end{aligned} \tag{4.48}$$

This concludes our proof. □

Lemma 4.12. *If n is the dimension of the vector space spanned by the discrete subgroup Z of X, then, for every $1 \le i \le n+1$, the set*

$$\mathcal{A}_i = \{A \subset X;\ A \textit{ is compact and } \operatorname{Cat}_{\pi(X)} \pi(A) \ge i\} \tag{4.49}$$

is nonempty. Moreover

$$\mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots \supset \mathcal{A}_{n+1}. \tag{4.50}$$

The proof of this result may be found in Mawhin and Willem [144].

Lemma 4.13. *For every $1 \leq i \leq n+1$, the set $\mathcal{A}_i$ becomes a complete metric space if it is endowed with the Hausdorff metric*

$$\delta(A,B) = \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\}. \tag{4.51}$$

The proof of this result may be found in Kuratowski [129].

Lemma 4.14. *If $f : X \to \mathbb{R}$ is continuous, then, for every $1 \leq i \leq n+1$, the mapping $\eta : \mathcal{A}_i \to \mathbb{R}$ defined by*

$$\eta(A) = \max_{x\in A} f(x) \tag{4.52}$$

is lower semicontinuous.

Proof. For any fixed i, let (A_n) be a sequence in $\mathcal{A}_i$ and $A \in \mathcal{A}_i$ such that $\delta(A_n, A) \to 0$.

For every $x \in A$ there exists a sequence (x_n) in X such that $x_n \in A_n$ and $x_n \to x$. Thus,

$$f(x) = \lim_{n\to\infty} f(x_n) \leq \liminf_{n\to\infty} \eta(A_n). \tag{4.53}$$

Since $x \in A$ is arbitrary, it follows that

$$\eta(A) \leq \liminf_{n\to\infty} \eta(A_n), \tag{4.54}$$

which concludes that the mapping η is lower semicontinuous. □

In what follows, $f : X \to \mathbb{R}$ is a locally Lipschitz functional which is Z-periodic and satisfies the condition $(\mathrm{PS})_Z$. Moreover, we assume that f is bounded from below. Let $\operatorname{Cr}(f,c)$ be the set of critical points of f having the real number c as corresponding critical value. Thus,

$$\operatorname{Cr}(f,c) = \{x \in X;\ f(x) = c \text{ and } \lambda(x) = 0\}. \tag{4.55}$$

If n is the dimension of the vector space spanned by the discrete group Z, then, for every $1 \leq i \leq n+1$, let

$$c_i = \inf_{A\in\mathcal{A}_i} \eta(A). \tag{4.56}$$

It follows by Lemma 4.12 and the boundedness from below of f that

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_{n+1} < +\infty. \tag{4.57}$$

The following abstract result is a Ljusternik-Schnirelmann-type theorem for locally Lipschitz Z-periodic functionals.

Theorem 4.15. *Under the above hypotheses, the functional f has at least $n+1$ distinct critical orbits.*

Proof. It is enough to show that if $1 \le i \le j \le n+1$ and $c_i = c_j = c$, then the set $\mathrm{Cr}(f,c)$ contains at least $j-i+1$ distinct critical orbits. Arguing by contradiction, let us assume that there exists $i \le j$ such that the set $\mathrm{Cr}(f,c)$ has $k \le j-i$ distinct critical orbits, generated by $x_1,\dots,x_k$. We first choose an open neighborhood of $\mathrm{Cr}(f,c)$ defined by

$$V_r = \bigcup_{l=1}^{k}\bigcup_{z\in Z} B(x_l+z,r). \tag{4.58}$$

Moreover we may assume that $r>0$ is chosen so that π restricted to the set $\overline{B}(x_l,2r)$ is one-to-one. This contradiction shows that, for every $1\le l\le k$,

$$\mathrm{Cat}_{\pi(X)}\,\pi\big(\overline{B}(x_l,2r)\big) = 1. \tag{4.59}$$

In the above arguments, $V_r = \varnothing$ if $k=0$.

Step 1. We prove that there exists $0<\varepsilon<\min\{1/4,r\}$ such that, for every $x\in[c-\varepsilon\le f\le c+\varepsilon]\setminus V_r$, we have

$$\lambda(x) > \sqrt{\varepsilon}. \tag{4.60}$$

Indeed, if not, there exists a sequence (x_m) in $X\setminus V_r$ such that, for every $m\ge 1$,

$$\begin{gathered} c-\frac{1}{m}\le f(x_m)\le c+\frac{1}{m},\\ \lambda(x_m)\le \frac{1}{\sqrt{m}}. \end{gathered} \tag{4.61}$$

Since f satisfies the condition $(\mathrm{PS})_Z$, passing eventually to a subsequence, we can assume that $\pi(x_m)\to\pi(x)$, for some $x\in V\setminus V_r$. Moreover, by the Z-periodicity property of f and λ, we can assume that $x_m\to x$. The continuity of f and the lower semicontinuity of λ imply $f(x)=0$ and $\lambda(x)=0$, contradiction, because $x\in V\setminus V_r$.

Step 2. For ε found above and taking into account the definition of c_j, there exists $A\in\mathcal{A}_j$ such that

$$\max_{x\in A} f(x) < c+\varepsilon^2. \tag{4.62}$$

Putting $B = A\setminus V_{2r}$ and applying Lemma 4.6, we find

$$\begin{aligned} j &\le \mathrm{Cat}_{\pi(X)}\,\pi(A) \le \mathrm{Cat}_{\pi(X)}\big(\pi(B)\cup(\overline{V}_{2r})\big)\\ &\le \mathrm{Cat}_{\pi(X)}\,\pi(B) + \mathrm{Cat}_{\pi(X)}\,\pi(\overline{V}_{2r}) \le \mathrm{Cat}_{\pi(X)}\,\pi(B)+k\\ &\le \mathrm{Cat}_{\pi(X)}\,\pi(B)+j-i. \end{aligned} \tag{4.63}$$

Therefore,

$$\mathrm{Cat}_{\pi(X)}\,\pi(B)\ge i, \tag{4.64}$$

that is, $B\in\mathcal{A}_i$.

Step 3. For ε and B as above, we apply Ekeland's variational principle to the functional η defined in Lemma 4.14. Thus, there exists $C \in \mathcal{A}_i$ such that, for every $D \in \mathcal{A}_i$, $D \neq C$,

$$\begin{gathered} \eta(C) \leq \eta(B) \leq \eta(A) \leq c + \varepsilon^2, \\ \delta(B, C) \leq \varepsilon, \end{gathered} \tag{4.65}$$

$$\eta(D) > \eta(C) - \varepsilon\delta(C, D). \tag{4.66}$$

Since $B \cap V_{2r} = \emptyset$ and $\delta(B, C) \leq \varepsilon < r$, we have $C \cap V_r = \emptyset$. In particular, the set $F = [f \geq c - \varepsilon]$ is contained in $[c - \varepsilon \leq f \leq c + \varepsilon]$ and $F \cap V_r = \emptyset$. Applying now Lemma 3.12 for $\varphi = \partial f$ defined on F, we find a continuous map $v : F \to X$ such that, for every $x \in F$ and $x^* \in \partial f(x)$,

$$\begin{gathered} \|v(x)\| \leq 1, \\ \langle x^*, v(x)\rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq \inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon, \end{gathered} \tag{4.67}$$

the last inequality being justified by relation (4.60). Thus, for every $x \in F$ and $x^* \in \partial f(x)$,

$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x)\rangle = -\min_{x^* \in \partial f(x)} \langle x^*, v(x)\rangle \leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon, \tag{4.68}$$

by the choice of ε.

By the upper semicontinuity of f^0 and the compactness of F there exists $\delta > 0$ such that, for every $x \in F$, $y \in X$, $\|y - x\| \leq \delta$, we have

$$f^0(y, -v(x)) < -\varepsilon. \tag{4.69}$$

Since $C \cap \operatorname{Cr}(f, c) = \emptyset$ and C is compact and $\operatorname{Cr}(f, c)$ is closed, there exists a continuous extension $w : X \to X$ of v such that the restriction of w to $\operatorname{Cr}(f, c)$ is the identic map and, for every $x \in X$, $\|w(x)\| \leq 1$.

Let $\alpha : X \to [0, 1]$ be a continuous function which is Z-periodic and such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \leq c - \varepsilon]$. Let $h : [0, 1] \times X \to X$ be the continuous map defined by

$$h(t, x) = x - t\delta\alpha(x)w(x). \tag{4.70}$$

If $D = h(1, C)$, it follows by Lemma 4.6 that

$$\operatorname{Cat}_{\pi(X)} \pi(D) \geq \operatorname{Cat}_{\pi(X)} \pi(C) \geq i, \tag{4.71}$$

which shows that $D \in \mathcal{A}_i$, because D is compact.

Step 4. By Lebourg's mean value theorem we deduce that for every $x \in X$, there is some $\theta \in (0, 1)$ such that

$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x)\rangle. \tag{4.72}$$

Thus, there exists $x^* \in \partial f(h(\theta, x))$ such that

$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta w(x)\rangle. \tag{4.73}$$

It follows now by (4.69) that if $x \in F$, then

$$f(h(1,x)) - f(h(0,x)) = \delta\alpha(x)\langle x^*, -w(x)\rangle \\ \leq \delta\alpha(x) f^0\langle x - \theta\delta\alpha(x)w(x), -v(x)\rangle \\ \leq -\varepsilon\delta\alpha(x). \tag{4.74}$$

Thus, for every $x \in C$,

$$f(h(1,x)) \leq f(x). \tag{4.75}$$

Let $x_0 \in C$ be such that $f(h(1,x_0)) = \eta(D)$. Therefore,

$$c \leq f(h(1,x_0)) \leq f(x_0). \tag{4.76}$$

By the definitions of α and F it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Thus, by (4.74), we have

$$f(h(1,x_0)) - f(x_0) \leq -\varepsilon\delta. \tag{4.77}$$

Hence

$$\eta(D) + \varepsilon\delta \leq f(x_0) \leq \eta(C). \tag{4.78}$$

Now, by the definition of D,

$$\delta(C,D) \leq \delta. \tag{4.79}$$

Thus

$$\eta(D) + \varepsilon\delta(C,D) \leq \eta(C), \tag{4.80}$$

that is, by (4.66), we find $C = D$, which contradicts the relation (4.78). □

4.4. A nonlinear eigenvalue problem arising in earthquake initiation

The main purpose of this section is to consider a nonlinear eigenvalue variational inequality arising in earthquake initiation and to establish, in the setting of the nonsmooth Ljusternik-Schnirelmann theory, the existence of infinitely many solutions. The main novelty in our framework is the presence of the convex cone of functions with nonnegative jump across an internal boundary which is composed of a finite number of bounded connected arcs.

Under some natural assumptions, we prove the existence of infinitely many solutions, as well as further properties of eigensolutions and eigenvalues. Since the associated energy functional is included neither in the theory of monotone operators nor in their Lipschitz perturbations, we employ the notion of lower subdifferential which is originally due to De Giorgi. Next, we are concerned with the study of the effect of a small nonsymmetric perturbation and we prove that the number of solutions of the perturbed problem becomes greater and greater if the perturbation tends to zero with respect to an appropriate topology.

Let Ω be a smooth, bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ containing a finite number of cuts. The internal boundary is denoted by Γ and the exterior one by Γ_d. On Γ we denote by $[\ ]$ the jump across Γ, that is, $[w] = w^+ - w^-$, and by $\partial_n = \nabla \cdot n$ the corresponding normal derivative with the unit normal n outwards the positive side. On the contact zone Γ, we have $[\partial_n w] = 0$. Denote by $\|\cdot\|$ the norm in the space

$$V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_d\} \tag{4.81}$$

and by $\Lambda_0 : L^2(\Omega) \to L^2(\Omega)^*$ and $\Lambda_1 : V \to V^*$ the duality isomorphisms defined by

$$\begin{aligned} \Lambda_0 u(v) &= \int_\Omega uv\,dx, \quad \text{for any } u, v \in L^2(\Omega), \\ \Lambda_1 u(v) &= \int_\Omega \nabla u \cdot \nabla v\,dx, \quad \text{for any } u, v \in V. \end{aligned} \tag{4.82}$$

Consider the Lipschitz map $\gamma = i \circ \eta : V \to L^2(\Gamma)$, where $\eta : V \to H^{1/2}(\Gamma)$ is the trace operator, $\eta(v) = [v]$ on Γ and $i : H^{1/2}(\Gamma) \to L^2(\Gamma)$ is the embedding operator. Standard arguments in the theory of Sobolev spaces imply that the space $L^2(\Gamma)$ is compactly embedded in V through the operator γ.

We are concerned with the following inequality problem:

$$\text{find } u \in K, \quad \lambda \in \mathbb{R} \text{ such that}$$

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Gamma j'\big(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\big)d\sigma + \lambda \int_\Omega u(v-u)dx \geq 0 \quad \forall v \in K, \tag{4.83}$$

where

$$j : \mathbb{R} \to \mathbb{R}; \quad j(t) = -\frac{\beta}{2}t^2 \tag{4.84}$$

and $j'(\cdot;\cdot)$ stands for the Gâteaux directional derivative.

Due to the homogeneity of (4.83), we can reformulate this problem in terms of a constrained inequality problem as follows. For any fixed $r > 0$, set

$$M = \left\{u \in V;\ \int_\Omega u^2 dx = r^2\right\}. \tag{4.85}$$

Then M is a smooth manifold in the Hilbert space V. We study the following problem:

$$\text{find } u \in K \cap M, \qquad \lambda \in \mathbb{R} \text{ such that}$$

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Gamma j'\big(\gamma(u(x)); \gamma(v(x)) - \gamma(u(x))\big)d\sigma + \lambda \int_\Omega u(v-u)dx \geq 0 \quad \forall v \in K. \tag{4.86}$$

We prove the following multiplicity result.

Theorem 4.16. *Problem* (4.86) *has infinitely many solutions* (u, λ) *and the set of eigenvalues* $\{\lambda\}$ *is bounded from above and its infimum equals to* $-\infty$. *Let* $\lambda_0 = \sup\{\lambda\}$. *Then there exists* u_0 *such that* (u_0, λ_0) *is a solution of* (4.86). *Moreover the function* $\beta \mapsto \lambda_0(\beta)$ *is convex and the following inequality holds:*

$$\int_\Omega |\nabla v|^2 dx + \lambda_0(\beta) \int_\Omega v^2 dx \geq \beta \int_\Gamma [v]^2 d\sigma \quad \forall v \in K. \tag{4.87}$$

Next, we study the effect of an arbitrary perturbation in problem (4.83). More precisely, we consider the following inequality problem:

$$\begin{aligned}
&\text{find } u_\varepsilon \in K, \quad \lambda_\varepsilon \in \mathbb{R} \text{ such that for any } v \in K \\
&\int_\Omega \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) dx \\
&\qquad + \int_\Gamma (j' + \varepsilon g')(\gamma(u_\varepsilon(x)); \gamma(v(x)) - \gamma(u_\varepsilon(x))) d\sigma \\
&\qquad + \lambda_\varepsilon \int_\Omega u_\varepsilon(v - u_\varepsilon) dx \geq 0,
\end{aligned} \tag{4.88}$$

where $\varepsilon > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with no symmetry hypothesis, but satisfying the growth assumption

$$\begin{aligned}
&\exists a > 0, \quad \exists 2 \leq p \leq \frac{2(N-1)}{N-2} \text{ such that } |g(t)| \leq a(1 + |t|^p), \quad \text{if } N \geq 3, \\
&\exists a > 0, \quad \exists 2 \leq p < +\infty \text{ such that } |g(t)| \leq a(1 + |t|^p), \quad \text{if } N = 2.
\end{aligned} \tag{4.89}$$

We prove that the number of solutions of problem (4.88) becomes greater and greater if the perturbation "tends" to zero. This is a very natural phenomenon that occurs often in concrete situations. We illustrate it with the following elementary example: consider on the real axis the equation $\sin x = 1/2$. This is a "symmetric" problem (due to the periodicity) with infinitely many solutions. Let us now consider an arbitrary nonsymmetric "small" perturbation of the above equation, say $\sin x = 1/2 + \varepsilon x^2$. This equation has finitely many solutions, for any $\varepsilon \neq 0$. However, the number of solutions of the perturbed equation tends to infinity if the perturbation (i.e., $|\varepsilon|$) becomes smaller and smaller.

More precisely, the following perturbation result holds.

Theorem 4.17. *For every positive integer n, there exists* $\varepsilon_n > 0$ *such that problem* (4.88) *has at least n distinct solutions* $(u_\varepsilon, \lambda_\varepsilon)$ *if* $\varepsilon < \varepsilon_n$. *There exists and is finite* $\lambda_{0\varepsilon} = \sup\{\lambda_\varepsilon\}$ *and there exists* $u_{0\varepsilon}$ *such that* $(u_{0\varepsilon}, \lambda_{0\varepsilon})$ *is a solution of* (4.88). *Moreover,* $\lambda_{0\varepsilon}$ *converges to* λ_0 *as* ε *tends to* 0, *where* λ_0 *is defined in Theorem 4.16.*

Before giving the proofs of the above results we recall some basic definitions and properties. We start with the following Sobolev-type inequality (we refer to [110] for a complete proof).

Proposition 4.18. *Let $2 \le \alpha \le 2(N-1)/(N-2)$ if $N \ge 3$ and $2 \le \alpha < +\infty$ if $N = 2$. Then for $\beta = [(\alpha-2)N+2]/(2\alpha)$ if $N \ge 3$ or if $N = 2$ and $\alpha = 2$ and for all $(\alpha-1)/\alpha < \beta < 1$ if $N = 2$ and $\alpha > 2$, there exists $C = C(\beta)$ such that for any $u \in V$,*

$$\left(\int_\Gamma |[u]|^\alpha d\sigma\right)^{1/\alpha} \le C\left(\int_\Omega u^2 dx\right)^{(1-\beta)/2}\left(\int_\Omega |\nabla u|^2 dx\right)^{\beta/2}. \tag{4.90}$$

An important role in our arguments in order to locate the solution of problem (4.86) will be played by the indicator function of M, that is,

$$I_M(u) = \begin{cases} 0 & \text{if } u \in M, \\ +\infty & \text{if } u \in V \setminus M. \end{cases} \tag{4.91}$$

Then I_M is lower semicontinuous. However, since the natural energy functional associated to problem (4.86) is neither smooth nor convex, it is necessary to introduce a more general concept of gradient.

We use the following notion of lower subdifferential which is due to De Giorgi, Marino, and Tosques [59]. The following definition agrees with the corresponding notions of gradient and critical point in the sense of Fréchet (for C^1 mappings), Clarke (for locally Lipschitz functionals) or in the sense of the convex analysis.

Definition 4.19. Let X be a Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary proper functional. Let $x \in D(f)$. The gradient of f at x is the (possibly empty) set

$$\partial^- f(x) = \left\{\xi \in X^*;\ \liminf_{y\to x} \frac{f(y) - f(x) - \xi(y-x)}{\|y-x\|} \ge 0\right\}. \tag{4.92}$$

An element $\xi \in \partial^- f(x)$ is called a lower subdifferential of f at x.

Accordingly, we say that $x \in D(f)$ is a critical (lower stationary) point of f if $0 \in \partial^- f(x)$.

Then $\partial^- f(x)$ is a convex set. If $\partial^- f(x) \ne \emptyset$ we denote by $\text{grad}^- f(x)$ the element of minimal norm of $\partial^- f(x)$, that is,

$$\text{grad}^- f(x) = \min\{\|\xi\|_{X^*};\ \xi \in \partial^- f(x)\}. \tag{4.93}$$

This notion plays a central role in the statement of our basic compactness condition.

Definition 4.20. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary functional. We say that $(x_n) \subset D(f)$ is a Palais-Smale sequence if

$$\sup_n |f(x_n)| < +\infty, \qquad \lim_{n\to\infty} \text{grad}^- f(x_n) = 0. \tag{4.94}$$

The functional f is said to satisfy the Palais-Smale condition provided that any Palais-Smale sequence is relatively compact.

Remark 4.21. (i) Definition 4.19 implies that if $g : X \to \mathbb{R}$ is Fréchet differentiable and $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper function, then

$$\partial^-(f+g)(x) = \{\xi + g'(x);\ \xi \in \partial^- f(x)\}, \tag{4.95}$$

for any $x \in D(f)$.

(ii) Similarly, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is an arbitrary proper functional and $g : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous, then

$$\partial^-(f+g)(x) = \{\xi + g'(x);\ \xi \in \partial^- f(x)\}, \tag{4.96}$$

for any $x \in D(f) \cap D(g)$.

As established in [43] by Chobanov, Marino, and Scolozzi,

$$\partial^- I_M(u) = \{\lambda \Lambda_0 u;\ \lambda \in \mathbb{R}\} \subset L^2(\Omega)^* \subset V^*, \quad \text{for any } u \in M. \tag{4.97}$$

Let (X,d) be a metric space. Consider $h : X \to \mathbb{R} \cup \{+\infty\}$ an arbitrary functional and set, as usually, $D(h) := \{u \in X;\ h(u) < +\infty\}$. We recall the following definitions which are due essentially to De Giorgi (see [59] by De Giorgi, Marino, and Tosques).

Definition 4.22. (i) For $u \in D(h)$ and $\rho > 0$, let $h_u(\rho) = \inf\{h(v);\ d(v,u) < \rho\}$. Then the number $-D_+h_u(0)$ is called the slope of h at u, where D_+ denotes the right lower derivative.

(ii) Let $I \subset \mathbb{R}$ be an arbitrary nontrivial interval and consider a curve $U : I \to X$. We say that U is a curve of maximal slope for h if the following properties hold true:

(a) U is continuous,
(b) $h \circ U(t) < +\infty$, for any $t \in I$,
(c) $d(U(t_2), U(t_1)) \le \int_{t_1}^{t_2} [D_+ h_{U(t)}(0)]^2 dt$, for any $t_1, t_2 \in I$, $t_1 < t_2$,
(d) $h \circ U(t_2) - h \circ U(t_1) \le -\int_{t_1}^{t_2} [D_+ h_{U(t)}(0)]^2 dt$, for any $t_1, t_2 \in I$, $t_1 < t_2$.

In what follows, X denotes a metric space, A is a subset of X and i stands for the inclusion map of A in X.

Definition 4.23. (i) A map $r : X \to A$ is said to be a retraction if it is continuous, surjective and $r_{|A} = Id$.

(ii) A retraction r is called a strong deformation retraction provided that there exists a homotopy $\zeta : X \times [0,1] \to X$ of $i \circ r$ and Id_X which satisfies the additional condition $\zeta(x,t) = \zeta(x,0)$, for any $(x,t) \in A \times [0,1]$.

(iii) The metric space X is said to be weakly locally contractible, if for every $u \in X$ there exists a neighborhood U of u contractible in X.

For every $a \in \mathbb{R}$, denote

$$f^a = \{u \in X : f(u) \le a\}, \tag{4.98}$$

where $f : X \to \mathbb{R}$ is a continuous function.

Definition 4.24. (i) Let $a, b \in \mathbb{R}$ with $a \leq b$. The pair (f^b, f^a) is said to be trivial provided that, for every neighborhood $[a', a'']$ of a and $[b', b'']$ of b, there exists some closed sets A and B such that $f^{a'} \subseteq A \subseteq f^{a''}$, $f^{b'} \subseteq B \subseteq f^{b''}$ and such that A is a strong deformation retraction of B.

(ii) A real number c is an essential value of f provided that, for every $\varepsilon > 0$ there exists $a, b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

The following property of essential values is due to Degiovanni and Lancelotti (see [60, Theorem 2.6]).

Proposition 4.25. *Let c be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \to \mathbb{R}$ with*

$$\sup\{|g(u) - f(u)| : u \in X\} < \delta \tag{4.99}$$

admits an essential value in $(c - \varepsilon, c + \varepsilon)$.

For every $n \geq 1$, define

$$\Gamma_n = \{S \subset S_r;\ S \subset \mathcal{F}, \gamma(S) \geq n\}, \tag{4.100}$$

where $\mathcal{F}$ is the class of closed symmetric subsets of the sphere S_r of radius r in a certain Banach space. We recall that $\gamma(S)$ stands for the Krasnoselski genus of $S \in \Gamma_n$, that is, $\gamma(S)$ is the smallest $k \in \mathbb{N} \cup \{+\infty\}$ for which there exists a continuous and odd map from S into $\mathbb{R}^k \setminus \{0\}$.

Define

$$E = F + G : V \longrightarrow \mathbb{R} \cup \{+\infty\}, \tag{4.101}$$

where

$$F(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases}$$
$$G(u) = -\frac{\beta}{2}\int_\Gamma [\gamma(u(x))]^2 d\sigma. \tag{4.102}$$

Then $E + I_M$ is lower semicontinuous.

The following auxiliary result shows that $E + I_M$ is the canonical energy functional associated to problem (4.86).

Proposition 4.26. *If (u, λ) is a solution of problem* (4.86), *then $0 \in \partial^-(E + I_M)(u)$. Conversely, let u be a critical point of $E + I_M$ and denote $\lambda = -2E(u)r^{-2}$. Then (u, λ) is a solution of problem* (4.86).

Proof. Let (u, λ) be a solution of problem (4.86). So, by the definition of the lower subdifferential,

$$-\lambda u \in \partial^- E(u). \tag{4.103}$$

On the other hand,

$$\partial^-(E+I_M)(u)=\partial^-E(u)+\partial^-I_M(u),\quad \text{for any } u\in K\cap M. \tag{4.104}$$

So, by relations (4.97) and (4.103), $0\in\partial^-(E+I_M)(u)$.

Conversely, let $0\in\partial^-(E+I_M)(u)$. Thus, by (4.97) and (4.104), there exists $\lambda\in\mathbb{R}$ such that (u,λ) is a solution of problem (4.86). If we put $v=0$ in (4.86) then we deduce $\lambda r^2\leq -2E(u)$ and for $v=2u$ we get $\lambda r^2\geq -2E(u)$, that is, $\lambda=-2E(u)r^{-2}$. □

The above result reduces our study to finding the critical points of $E+I_M$. In order to estimate the number of lower stationary points of this functional we will apply a nonsmooth version of the Ljusternik-Schnirelmann theorem. For this purpose we need some preliminary results.

We first observe that a direct argument combined with Proposition 4.26 show that problem (4.86) has at least one solution. Indeed, the associated energy functional is bounded from below. This follows directly by our basic inequality (4.90) since

$$(E+I_M)(u)\geq\frac{1}{2}\|u\|^2-|\beta|\cdot||[u]||^2_{L^2(\Gamma)}\geq\frac{1}{2}\|u\|^2-C\|u\|\geq C_0, \tag{4.105}$$

for any $u\in V$. So, by standard minimization arguments based on the compactness of the embedding $i\circ\eta:V\to L^2(\Gamma)$ we deduce that there exists a global minimum point $u_0\in K\cap M$ of $E+I_M$. Let $\lambda_0=-2E(u_0)/r^2$. Hence $0\in\partial^-(E+I_M)(u_0)$ and (u_0,λ_0) is a solution of problem (4.86). Since for any eigenvalue λ there exists $u\in K$ such that $\lambda=-2E(u)r^{-2}$ we deduce that $\lambda_0=\sup\{\lambda\}$.

The next step in the proof of Theorem 2.3 is the following.

Proposition 4.27. *The functional $E+I_M$ satisfies the Palais-Smale condition.*

Proof. Let (u_n) be an arbitrary Palais-Smale sequence of $E+I_M$. So, by (4.105), (u_n) is bounded in V. Thus, by the Rellich-Kondratchov theorem and passing eventually at a subsequence,

$$u_n\rightharpoonup u\quad\text{weakly in } V, \tag{4.106}$$

$$u_n\longrightarrow u\quad\text{strongly in } L^2(\Omega), \tag{4.107}$$

$$u_n\longrightarrow u\quad\text{strongly in } L^2(\Gamma). \tag{4.108}$$

In particular, it follows that $u\in K\cap M$.

Using now the second information contained in the statement of the Palais-Smale condition and applying (4.97), we obtain a sequence (λ_n) of real numbers such that

$$\lim_{n\to\infty}||E'(u_n)+\lambda_n\Lambda_0u_n||_{V^*}=0. \tag{4.109}$$

On the other hand, by the compact embeddings $V\subset L^2(\Omega)$ and $V\subset L^2(\Gamma)$, and using (4.106)–(4.108) it follows that

$$E'(u_n)\longrightarrow E'(u),\qquad \Lambda_1u_n\longrightarrow\Lambda_1u\quad\text{in } V^*. \tag{4.110}$$

So, by (4.109), the sequence (λ_n) is bounded. Hence we can assume that, up to a subsequence, $\lambda_n \to \lambda$ as $n \to \infty$. Therefore, $0 \in \partial^-(E + I_M)(u)$.

From (4.106) we get $\|u\| \le \liminf_{n\to\infty} \|u_n\|$, hence it follows that for concluding the proof it is enough to show that

$$\|u\| \ge \limsup_{n\to\infty} \|u_n\|. \tag{4.111}$$

But, since F is convex,

$$F(u) \ge F(u_n) + F'(u_n)(u - u_n). \tag{4.112}$$

It follows that

$$\begin{aligned} E(u) &= F(u) + G(u) \ge \limsup_{n\to\infty} \left(F(u_n) + F'(u_n)(u - u_n) + G(u_n)\right) \\ &= \limsup_{n\to\infty} \left(F(u_n) + F'(u_n)(u - u_n) + G(u_n) + G'(u_n)(u - u_n)\right) \\ &= \limsup_{n\to\infty} \left(F(u_n) + F'(u_n)(u - u_n)\right) + \lim_{n\to\infty} G(u_n). \end{aligned} \tag{4.113}$$

Using now $\lambda_n \to \lambda$ combined with (4.106)–(4.109), relation (4.113) yields

$$E(u) \ge \limsup_{n\to\infty} F(u_n) + G(u). \tag{4.114}$$

This inequality implies directly our claim (4.111), so the proof is completed. □

Due to the symmetry of our problem (4.86), we can extend our study to the symmetric cone $(-K)$. More precisely, if (u, λ) is a solution of (4.86) then $u_0 := -u \in (-K) \cap M$ satisfies

$$\begin{aligned} &\int_\Omega \nabla u_0 \cdot \nabla(v - u_0)\,dx \\ &\quad + \int_\Gamma j'\left(\gamma(u_0(x)); \gamma(v(x)) - \gamma(u_0(x))\right)d\sigma \\ &\quad + \lambda \int_\Omega u_0(v - u_0)\,dx \ge 0, \quad \forall v \in (-K). \end{aligned} \tag{4.115}$$

This means that we can extend the energy functional associated to problem (4.86) to the symmetric set $\widetilde{K} := K \cup (-K)$. We put, by definition,

$$\widetilde{E}(u) = \begin{cases} E(u) & \text{if } u \in K, \\ E(-u) & \text{if } u \in (-K), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.116}$$

We are now interested in finding the lower stationary points of the extended energy functional $J := \widetilde{E} + I_M$.

We endow the set $\widetilde{K} \cap M$ with the graph metric of $\widetilde{E}$ defined by

$$d(u,v) = \|u - v\| + \left|\widetilde{E}(u) - \widetilde{E}(v)\right|, \quad \text{for any } u, v \in \widetilde{K} \cap M. \tag{4.117}$$

Denote by χ the metric space $(\widetilde{K} \cap M, d)$.

We are now in position to state the basic abstract result that we will apply for concluding the proof of Theorem 4.16. More precisely, we use the following nonsmooth variant of the Ljusternik-Schnirelmann theory that we reformulate in terms of our energy functional J.

Theorem 4.28 (Marino and Scolozzi [141]). *Assume that J satisfies the following properties:*

(i) *J is bounded from below;*
(ii) *J satisfies the Palais-Smale condition;*
(iii) *for any lower stationary point u of J there exists a neighborhood of u in χ which is contractible in χ;*
(iv) *there exists $\Theta : (\widetilde{K} \cap M) \times [0, \infty) \to \widetilde{K} \cap M$ such that $\Theta(\cdot, 0) = \mathrm{Id}$, $\Theta(u, \cdot)$ is a curve of maximal slope for J (with respect to the usual metric in V) and, moreover, the mapping $\Theta : \chi \times [0, \infty) \to \chi$ is continuous.*

Then J has at least $\mathrm{Cat}_\chi(\widetilde{K} \cap M)$ lower stationary points.

Moreover, if $\mathrm{Cat}_\chi(\widetilde{K} \cap M) = +\infty$, then J does not have a maximum and

$$\sup\left\{J(u);\ u \in \widetilde{K} \cap M, 0 \in \partial^- J(u)\right\} = \sup\left\{J(u);\ u \in \widetilde{K} \cap M\right\}. \tag{4.118}$$

We have already proved (i) and (ii). Property (iii) is proved in a more general framework in De Giorgi, Marino, and Tosques [59], while (iv) is deduced in Chobanov, Marino and Scolozzi [43]. So, using Theorem 4.28, it follows that for concluding the proof of Theorem 4.16 it remains to prove the following result.

Proposition 4.29. *One has*

$$\mathrm{Cat}_\chi(\widetilde{K} \cap M) = +\infty. \tag{4.119}$$

Proof. Fix $\psi \in K \setminus \{0\}$ such that $\|\psi\|_{L^2(\Omega)} > r$ and let $(e_n)_{n\geq 1} \subset V$ be an orthonormal basis of $L^2(\Omega)$. Fix arbitrarily an integer $n \geq 1$ and denote

$$M^{(n)} = \left\{\sum_{i=1}^{n} \alpha_i e_i;\ \sum_{i=1}^{n} \alpha_i^2 = r^2\right\}. \tag{4.120}$$

As usually, we denote $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$, for any real number a. Define the mapping $\varphi_1 : M^{(n)} \times [0,1] \to V \setminus \{0\}$ by

$$\varphi_1(u,t) = (1-t)\left[(u-\psi)^+ - (u+\psi)^-\right] + P_K\left(\min\left\{\max(u, -\psi), \psi\right\}\right), \tag{4.121}$$

where P_K denotes the canonical projection onto K. Then

$$\varphi_1(u,1) \in K, \qquad \left\|\varphi_1(u,1)\right\|_{L^2} \leq \|u\|_{L^2} \leq r. \tag{4.122}$$

We also define $\varphi_2 : (\widetilde{K} \setminus \{0\}) \times [0,1) \to \widetilde{K} \setminus \{0\}$ by

$$\varphi_2(u,t) = \min\left[\max\left(\frac{1}{1-t}u, -\psi\right), \psi\right]. \tag{4.123}$$

Fix arbitrarily $u \in \varphi_1(M^{(n)}, 1)$. Then

$$\lim_{t \nearrow 1} \left\|\varphi_2(u,t)\right\|_{L^2} = \|\psi\|_{L^2} > r. \tag{4.124}$$

The compactness of $\varphi_1(M^{(n)}, 1)$ implies that there exists $t_0 \in (0,1)$ such that

$$||\varphi_2(u,t)||_{L^2} > r \quad \forall t \in [t_0, 1),\ \forall u \in \varphi_1(M^{(n)}, 1). \tag{4.125}$$

Let P be the canonical projection of V onto the closed ball of radius r in $L^2(\Omega)$ centered at the origin. Define the map $\Phi : M^{(n)} \times [0, 1 + t_0] \to V \setminus \{0\}$ by

$$\Phi(u,t) = \begin{cases} \varphi_1(u,t) & \text{if } (u,t) \in M^{(n)} \times [0,1], \\ P(\varphi_2(\varphi_1(u,1), t-1)) & \text{if } (u,t) \in M^{(n)} \times [0, 1+t_0]. \end{cases} \tag{4.126}$$

Then $\Phi(u,0) = 0$ and $\Phi(u, 1+t_0) \in M$. Since $\Phi(\cdot, t)$ is odd and continuous from $L^2(\Omega)$ in the L^2-topology, it follows by Lemma 4.6 that

$$n \le \mathrm{Cat}_{L^2}(M^{(n)}) \le \mathrm{Cat}_{L^2}(\Phi(M^{(n)}, 1+t_0)) \le \mathrm{Cat}_{H_0^1}(\Phi(M^{(n)}, 1+t_0)). \tag{4.127}$$

Since the set $\Phi(M^{(n)}, 1+t_0)$ is compact in V and the topology of χ is stronger than the H_0^1-topology, we obtain

$$n \le \mathrm{Cat}_{H_0^1}(\Phi(M^{(n)}, 1+t_0)) \le \mathrm{Cat}_{\chi}(\Phi(M^{(n)}, 1+t_0)) \le \mathrm{Cat}_{\chi}(\widetilde{K} \cap M). \tag{4.128}$$

This completes the proof of Proposition 4.29. □

Proof of Theorem 4.16 completed. Until now, using Theorem 4.28, we have established that problem (4.86) admits infinitely many solutions (u, λ). We first observe that the set of eigenvalues is bounded from above. Indeed, if (u, λ) is a solution of our problem then choosing $v = 0$ in (4.86) and using (4.90), it follows that

$$\lambda r^2 \le -2\|u\|^2 + \frac{\beta}{2}\|u\|^2_{L^2(\Gamma)} \le C, \tag{4.129}$$

where C does not depend on u.

It remains to prove that

$$\inf\{\lambda; \lambda \text{ is an eigenvalue of } (4.86)\} = -\infty. \tag{4.130}$$

For this purpose, it is sufficient to show that

$$\sup\{J(u);\ u \in \widetilde{K} \cap M\} = +\infty. \tag{4.131}$$

But this follows directly from (4.90) and

$$\sup_{u\in\widetilde{K}\cap M}\int_\Omega |\nabla u|^2 dx = +\infty. \tag{4.132}$$

In order to prove the last part of the theorem we remark that $-\lambda_0$, as a function of β, is the upper bound of a family of affine functions

$$-\lambda_0(\beta) = \inf_{v\in K\cap M}\frac{1}{r^2}\left\{\int_\Omega |\nabla v|^2 dx - \beta\int_\Gamma [v]^2 d\sigma\right\}, \tag{4.133}$$

hence it is a concave function. Thus, the mapping $\beta \mapsto \lambda_0(\beta)$ is convex and the proof of Theorem 4.16 is completed. □

Proof of Theorem 4.17. We will establish the multiplicity result with respect to a prescribed level of energy. More precisely, let us fix $r > 0$. Consider the manifold

$$N = \left\{u \in V;\ \int_\Gamma [u]^p d\sigma = r^p\right\}, \tag{4.134}$$

where p is as in (4.89).

We reformulate problem (4.88) as follows:

$$\begin{gathered}\text{find } u_\varepsilon \in K\cap N,\quad \lambda_\varepsilon \in \mathbb{R} \text{ such that for any } v \in K,\\ \int_\Omega \nabla u_\varepsilon \cdot \nabla(v-u_\varepsilon)dx \\ +\int_\Gamma (j'+\varepsilon g')(\gamma(u_\varepsilon(x));\gamma(v(x))-\gamma(u_\varepsilon(x)))d\sigma \\ +\lambda_\varepsilon\int_\Omega u_\varepsilon(v-u_\varepsilon)dx \geq 0.\end{gathered} \tag{4.135}$$

□

We start with the following preliminary result.

Lemma 4.30. *There exists a sequence (b_n) of essential values of E such that $b_n \to +\infty$ as $n \to \infty$.*

Proof. For any $n \geq 1$, set $a_n = \inf_{S\in\Gamma_n}\sup_{u\in S}E(u)$, where Γ_n is the family of compact subsets of $K\cap N$ of the form $\phi(S^{n-1})$, with $\phi : S^{n-1} \to K\cap N$ continuous and odd. The function E restricted to $K\cap N$ is continuous, even and bounded from below. So, by Theorem 2.12 in [60], it is sufficient to prove that $a_n \to +\infty$ as $n \to \infty$. But, by Proposition 4.27, the functional E restricted to $K\cap N$ satisfies the Palais-Smale condition. So, taking into account Theorem 3.5 in [51] and Theorem 3.9 in [60], we deduce that the set E^c has finite genus for any $c \in \mathbb{R}$. Using now the definition of the genus combined with the fact that $K\cap N$ is a weakly locally contractible metric space, we deduce that $a_n \to +\infty$. This completes our proof. □

The canonical energy associated to problem (4.135) is the functional J restricted to $K \cap N$, where $J = E + \Phi$ and Φ is defined by

$$\Phi(u) = \varepsilon \int_\Gamma g\big(\gamma(u(x))\big)d\sigma. \tag{4.136}$$

A straightforward computation with the same arguments as in the proof of Proposition 4.26 shows that if u is a lower stationary point of J then there exists $\lambda \in \mathbb{R}$ such that (u,λ) is a solution of problem (4.135). In virtue of this result, it is sufficient for concluding the proof of Theorem 4.17 to show that the functional J has at least n distinct critical values, provided that $\varepsilon > 0$ is sufficiently small. We first prove that J is a small perturbation of E. More precisely, we have

Lemma 4.31. *For every $\eta > 0$, there exists $\delta = \delta_\eta > 0$ such that* $\sup_{u \in K\cap N} |J(u) - E(u)| \leq \eta$, *provided that $\varepsilon \leq \delta$.*

Proof. We have

$$\big|J(u) - E(u)\big| = \big|\Phi(u)\big| \leq \varepsilon \int_\Gamma \big|g\big(\gamma(u(x))\big)\big|d\sigma. \tag{4.137}$$

So, by (4.89) and Proposition 4.18,

$$\big|J(u) - E(u)\big| \leq \varepsilon a \int_\Gamma \big(1 + [u(x)]^p\big)d\sigma \leq C\varepsilon \leq \eta, \tag{4.138}$$

if ε is sufficiently small.

By Lemma 4.30, there exists a sequence (b_n) of essential values of $E_{|K\cap N}$ such that $b_n \to +\infty$. Without loss of generality we can assume that $b_i < b_j$ if $i < j$. Fix an integer $n \geq 1$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < 1/2 \min_{2\leq i\leq n}(b_i - b_{i-1})$. Applying now Proposition 4.25, we obtain that for any $1 \leq j \leq n$, there exists $\eta_j > 0$ such that if $\sup_{K\cap N} |J(u) - E(u)| < \eta_j$ then $J_{|K\cap N}$ has an essential value $c_j \in (b_j - \varepsilon_0, b_j + \varepsilon_0)$. So, by Lemma 4.31 applied for $\eta = \min\{\eta_1, \dots, \eta_n\}$, there exists $\delta_n > 0$ such that $\sup_{K\cap N} |J(u) - E(u)| < \eta$, provided that $\varepsilon \leq \delta_n$. This shows that the energy functional J has at least n distinct essential values $c_1, \dots, c_n$ in $(b_1 - \varepsilon_0, b_n + \varepsilon_0)$.

The next step consists in showing that $c_1, \dots, c_n$ are critical values of $J_{|K\cap N}$. Arguing by contradiction, let us suppose that c_j is not a critical value of $J_{|K\cap N}$. We show in what follows that

(A_1) There exists $\bar{\delta} > 0$ such that $J_{|K\cap N}$ has no critical value in $(c_j - \bar{\delta}, c_j + \bar{\delta})$.

(A_2) For every $a, b \in (c_j - \bar{\delta}, c_j + \bar{\delta})$ with $a < b$, the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial.

Suppose, by contradiction, that (A_1) is no valid. Then there exists a sequence (d_k) of critical values of $J_{|K\cap N}$ with $d_k \to c_j$ as $k \to \infty$. Since d_k is a critical value, it follows that there exists $u_k \in K \cap N$ such that

$$J(u_k) = d_k, \qquad 0 \in \partial^- J(u_k). \tag{4.139}$$

Using now the fact that J satisfies the Palais-Smale condition at the level c_j, it follows that, up to a subsequence, (u_k) converges to some $u \in K \cap N$ as $k \to \infty$. So, by the continuity of J and the lower semicontinuity of grad$J(\cdot)$, we obtain $J(u) = c_j$ and $0 \in \partial^- J(u)$, which contradicts the initial assumption on c_j.

We now prove assertion (A_2). For this purpose we apply the noncritical point theorem (see [51, Theorem 2.15]). So, there exists a continuous map $\chi : (K \cap N) \times [0,1] \to K \cap N$ such that

$$\begin{gathered} \chi(u,0) = u, \quad J\big(\chi(u,t)\big) \le J(u), \\ J(u) \le b \Longrightarrow J\big(\chi(u,1)\big) \le a, \quad J(u) \le a \Longrightarrow \chi(u,t) = u. \end{gathered} \tag{4.140}$$

Define the map $\rho : J^b_{|K\cap N} \to J^a_{|K\cap N}$ by $\rho(u) = \chi(u,1)$. From (4.140) we obtain that ρ is well defined and it is a retraction. Set

$$\mathcal{J} : J^b_{|K\cap N} \times [0,1] \longrightarrow J^b_{|K\cap N}, \qquad \mathcal{J}(u,t) = \chi(u,t). \tag{4.141}$$

The definition of $\mathcal{J}$ implies that, for every $u \in J^b_{|K\cap N}$,

$$\mathcal{J}(u,0) = u, \qquad \mathcal{J}(u,1) = \rho(u) \tag{4.142}$$

and, for any $(u,t) \in J^a_{|K\cap N} \times [0,1]$,

$$\mathcal{J}(u,t) = \mathcal{J}(u,0). \tag{4.143}$$

From (4.142) and (4.143) it follows that $\mathcal{J}$ is $J^a_{|K\cap N}$-homotopic to the identity of $J^a_{|K\cap N}$, that is, $\mathcal{J}$ is a strong deformation retraction, so the pair $(J^b_{|K\cap N}, J^a_{|K\cap N})$ is trivial. Assertions (A_1), (A_2), and Definition 4.24(ii) show that c_j is not an essential value of $J_{|K\cap N}$. This contradiction concludes our proof. □

4.5. Bibliographical notes

Min-max methods (or the calculus of variations in the large) were introduced by Birkhoff [26] and later developed by Ljusternik and Schnirelmann [140] in the first half of the last century. A basic tool to evaluate the size of a set in a topological space is the so-called Ljusternik-Schnirelmann category. The multiplicity result stated in Theorem 4.15 (a Ljusternik-Schnirelmann type theorem for periodic functionals) is due to Mawhin and Willem [144] for C^1-functionals and to Rădulescu [192] in the locally Lipschitz setting developed in this section. Theorems 4.16 and 4.17 are due to Rădulescu [196] (see also Ionescu and Rădulescu [111] for further developments).

5 Multivalued elliptic problems

Inequality is the cause of all local movements.
Leonardo da Vinci

5.1. Introduction

The field of multivalued problems has seen a considerable development in mathematics and applied sciences in a remarkably short time. This is mainly due to the fact that the mathematical tools used in the qualitative analysis of inequality problems are very efficient in the understanding of wide classes of concrete problems arising in applications.

In this chapter, we are concerned with various multivalued elliptic problems in a variational setting. The theory of variational inequalities appeared in the 1960s in connection with the notion of subdifferential in the sense of convex analysis. For many details we refer to the pioneering works by Brezis [29], Ekeland and Temam [77], and Kinderlehrer and Stampacchia [121]. All the inequality problems treated to the beginning of 1980s were related to convex energy functionals and therefore strictly connected to monotonicity: for instance, only monotone (possibly multivalued) boundary conditions and stress-strain laws could be studied.

Nonconvex inequality problems first appeared in the setting of global analysis and were related to the subdifferential introduced by De Giorgi, Marino, and Tosques [59]. In the setting of continuum mechanics, P. D. Panagiotopoulos started the study of nonconvex and nonsmooth potentials by using Clarke's subdifferential for locally Lipschitz functionals. Due to the lack of convexity, new types of inequality problems, called hemivariational inequalities, have been generated. Roughly speaking, mechanical problems involving nonmonotone stress-strain laws or boundary conditions derived by nonconvex superpotentials lead to hemivariational inequalities. We refer to the monographs by Naniewicz and Panagiotopoulos [164] and Panagiotopoulos [175] for the main aspects of this theory.

A typical feature of nonconvex problems is that, while in the convex case the stationary variational inequalities give rise to minimization problems for the potential or for the energy, in the nonconvex case the problem of the stationarity of the potential emerges and therefore it becomes reasonable to expect results also in the line of critical point theory.

For hemivariational inequalities, the associated energy functional E is typically of the form $E = E_0 + E_1$, where E_0 is the principal part satisfying some standard coerciveness condition and f_1 is locally Lipschitz. In such a setting, the main abstract tool is constituted by the nonsmooth critical point theory for locally Lipschitz functionals.

5.2. A multivalued version of the Brezis-Nirenberg problem

Truth lies neither in the narrow frame of doctrines nor outside them. Attempt to find it must carry us beyond all boundaries and dogmatic limitations.
Stelian Mihalaş (1975)

Let Ω be an open bounded set with the boundary sufficiently smooth in $\mathbb{R}^N$. Let g be a measurable function defined on $\Omega \times \mathbb{R}$ and such that

$$|g(x,t)| \le C(1+|t|^p), \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}, \tag{5.1}$$

where C is a positive constant and $1 \le p < (N+2)/(N-2)$ if $N \ge 3$, while $1 \le p < \infty$ if $N = 1, 2$.

Define the functional $\psi : L^{p+1}(\Omega) \to \mathbb{R}$ by

$$\psi(u) = \int_\Omega \int_0^{u(x)} g(x,t)\,dt\,dx. \tag{5.2}$$

We first prove that ψ is a locally Lipschitz map. Indeed the growth condition (5.1) and Hölder's inequality yield

$$|\psi(u) - \psi(v)| \le C'\left(|\Omega|^{p/(p+1)} + \max_{w \in U} \|w\|^{p/(p+1)}_{L^{p+1}(\Omega)}\right) \cdot \|u - v\|_{L^{p+1}(\Omega)}, \tag{5.3}$$

where U is an open ball containing u and v.

Set

$$\begin{aligned} \underline{g}(x,t) &= \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf} \{g(x,s); |t-s| < \varepsilon\}, \\ \overline{g}(x,t) &= \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup} \{g(x,s);\ |t-s| < \varepsilon\}. \end{aligned} \tag{5.4}$$

Lemma 5.1. *The mappings $\underline{g}$ and $\overline{g}$ are measurable.*

Proof. Observe that

$$\begin{aligned} \overline{g}(x,t) &= \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup} \{g(x,s);\ s \in [t-\varepsilon, t+\varepsilon]\} \\ &= \lim_{n \to \infty} \operatorname{ess\,sup} \left\{g(x,s);\ s \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right]\right\}. \end{aligned} \tag{5.5}$$

Replacing, locally, the map g by $g + M$, for M large enough, we may assume that $g \geq 0$. It follows that

$$\begin{aligned}\overline{g}(x,t) &= \lim_{n\to\infty} \|g(x,\cdot)\|_{L^\infty([t-1/n,t+1/n])} \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \|g(x,\cdot)\|_{L^m([t-1/n,t+1/n])} \\ &= \lim_{n\to\infty}\lim_{m\to\infty} g_{m,n}(x,t),\end{aligned} \tag{5.6}$$

where

$$g_{m,n}(x,t) = \left(\int_{t-1/n}^{t+1/n} |g(x,s)|^m ds\right)^{1/m} \tag{5.7}$$

Thus, it is sufficient to prove that if $h \in L^\infty_{\text{loc}}(\Omega\times\mathbb{R})$ and $a > 0$, then the mapping

$$k : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}, \qquad k(x,t) = \int_{t-a}^{t+a} h(x,s)ds \tag{5.8}$$

is measurable and to apply this result for $h(x,s) = |g(x,s)|^m$ and $a = 1/n$.

Observe that, assuming we have already proved that

$$\Omega\times\mathbb{R}\times\mathbb{R} \ni (x,s,t) \stackrel{l}{\longmapsto} h(x,s+t) \tag{5.9}$$

is measurable, then the conclusion follows by

$$k(x,t) = \int_{-a}^{a} l(x,s,t)ds \tag{5.10}$$

and Fubini's theorem applied to the locally integrable function l (the local integrability follows from its local boundedness). So, it is sufficient to justify the measurability of the mapping l. In order to do this, it is enough to prove that reciprocal images of Borel sets (resp., negligible sets) through the function

$$\Omega\times\mathbb{R}\times\mathbb{R} \ni (x,t,s) \stackrel{\omega}{\longmapsto} (x,s+t) \in \Omega\times\mathbb{R} \tag{5.11}$$

are Borel sets (resp., negligible). The first condition is, obviously, fulfilled. For the second, let A be a measurable set of null measure in $\Omega\times\mathbb{R}$. Consider the map

$$\mathbb{R}^N\times\mathbb{R}\times\mathbb{R} \ni (x,t,s) \stackrel{\eta}{\longmapsto} (x,s+t,t) \in \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}. \tag{5.12}$$

We observe that η is invertible and of class C^1. Moreover,

$$\eta(\omega^{-1}(A)) \subset A\times\mathbb{R}, \tag{5.13}$$

which is negligible. So, $\omega^{-1}(A)$ which is negligible, too. □

Let $G : \Omega\times\mathbb{R} \to \mathbb{R}$ be defined by

$$G(x,t) = \int_0^t g(x,s)ds. \tag{5.14}$$

Lemma 5.2. *Let g be a locally bounded measurable function, defined on $\Omega \times \mathbb{R}$ and $\underline{g}$, $\overline{g}$ as above.*

Then the Clarke subdifferential of G with respect to t is given by

$$\partial_t G(x,t) = \left[\underline{g}(x,t), \overline{g}(x,t)\right], \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}. \tag{5.15}$$

The condition "a.e." can be removed if, for every x, the mapping $t \mapsto g(x,t)$ is measurable and locally bounded.

Proof. We show that we have equality on the set

$$\{(x,t) \in \Omega \times \mathbb{R};\ g(x,\cdot) \text{ is locally bounded and measurable}\}. \tag{5.16}$$

In order to prove our result, it is enough to consider mappings g which do not depend on x. For this aim the equality that we have to prove is equivalent to

$$G^0(t;1) = \overline{g}(t), \qquad G^0(t;-1) = \underline{g}(t). \tag{5.17}$$

Examining the definitions of $G^0, \overline{g}$, and $\underline{g}$, it follows that

$$\underline{g}(t) = -(\overline{-g})(t), \qquad G^0(t,-1) = -(-G)^0(t,1). \tag{5.18}$$

So, the second equality appearing in (5.17) is equivalent to the first one.

The inequality $G^0(t,1) \leq \overline{g}(t)$ is proved in Chang [41]. For the reversed inequality, we assume by contradiction that there exists $\varepsilon > 0$ such that

$$G^0(t,1) = \overline{g}(t) - \varepsilon. \tag{5.19}$$

Let $\delta > 0$ be such that

$$\frac{G(\tau+\lambda) - G(\tau)}{\lambda} < \overline{g}(t) - \frac{\varepsilon}{2}, \tag{5.20}$$

if $0 < |\tau - t| < \delta$ and $0 < \lambda < \delta$. Then

$$\frac{1}{\lambda}\int_\tau^{\tau+\lambda} g(s)ds < \overline{g}(t) - \frac{\varepsilon}{2}, \quad \text{if } |\tau - t| < \delta,\ \lambda > 0. \tag{5.21}$$

We now justify the existence of some $\lambda_n \searrow 0$ such that

$$\frac{1}{\lambda_n}\int_\tau^{\tau+\lambda_n} g(s)ds \longrightarrow g(\tau), \quad \text{a.e. } \tau \in (t-\delta, t+\delta). \tag{5.22}$$

Assume, for the moment, that (5.22) has already been proved. Then, by relations (5.21) and (5.22) it follows that for every $\tau \in (t-\delta, t+\delta)$,

$$g(\tau) \leq \overline{g}(t) - \frac{\varepsilon}{2}. \tag{5.23}$$

Thus we get the contradiction

$$\overline{g}(t) \leq \operatorname{ess\,sup}\{g(s);\ s \in [t-\delta, t+\delta]\} \leq \overline{g}(t) - \frac{\varepsilon}{2}. \tag{5.24}$$

For concluding the proof it is sufficient to prove (5.22). Observe that we can "cut off" the mapping g, in order to have $g \in L^\infty \cap L^1$. Then (5.22) is nothing else that the classical result

$$T_\lambda \longrightarrow \mathrm{Id}_{L^1(\mathbb{R})}, \quad \text{if } \lambda \searrow 0 \tag{5.25}$$

in $\mathcal{L}(L^1(\mathbb{R}))$, where

$$T_\lambda u(t) = \frac{1}{\lambda}\int_t^{t+\lambda} u(s)ds, \tag{5.26}$$

for $\lambda > 0$, $t \in \mathbb{R}$, $u \in L^1(\mathbb{R})$.

Indeed, we observe easily that T_λ is linear and continuous in in $L^1(\mathbb{R})$ and that

$$\lim_{\lambda\searrow 0} T_\lambda u = u \quad \text{in } \mathcal{D}(\mathbb{R}), \tag{5.27}$$

for $u \in \mathcal{D}(\mathbb{R})$. The relation (5.25) follows now by a density argument. □

Returning to our problem, it follows by [41, Theorem 2.1] by Chang that

$$\partial\psi_{|H_0^1(\Omega)}(u) \subset \partial\psi(u). \tag{5.28}$$

For obtaining further information concerning $\partial\psi$, we need the following refinement of Theorem 2.1 in [41].

Theorem 5.3. *If $u \in L^{p+1}(\Omega)$ then*

$$\partial\psi(u)(x) \subset \left[\underline{g}(x,u(x)),\overline{g}(x,u(x))\right], \quad \text{a.e. } x \in \Omega, \tag{5.29}$$

in the sense that if $w \in \partial\psi(u)$, then

$$\underline{g}(x,u(x)) \le w(x) \le \overline{g}(x,u(x)), \quad \text{a.e. } x \in \Omega. \tag{5.30}$$

Proof. Let h be a Borel function such that $h = g$ a.e. in $\Omega \times \mathbb{R}$. Then

$$A = \Omega \setminus \{x \in \Omega;\ h(x,t) = g(x,t) \text{ a.e. } t \in \mathbb{R}\} \tag{5.31}$$

is a negligible set. Thus

$$B = \{x \in \Omega;\ \text{there exists } t \in \mathbb{R} \text{ such that } \underline{g}(x,t) \ne \underline{h}(x,t)\} \tag{5.32}$$

is a negligible set. A similar reasoning may be done for $\overline{g}$ and $\overline{h}$.

It follows that we can assume g is a Borel function.

Lemma 5.4. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. Then $\overline{g}$ is a Borel function.*

Proof of Lemma 5.4. Since the restriction is local, we may assume that g is nonnegative and bounded by 1. Since

$$g = \lim_{n\to\infty}\ \lim_{m\to\infty} g_{m,n}, \tag{5.33}$$

where

$$g_{m,n}(x,t) = \left(\int_{t-1/n}^{t+1/n} |g^m(x,s)|\,ds \right)^{1/m}, \tag{5.34}$$

it is enough to show that $g_{m,n}$ is a Borel function. □

Set

$$\mathcal{M} = \{g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R};\ |g| \leq 1,\ g \text{ is borelian}\},$$
$$\mathcal{N} = \{g \in \mathcal{M};\ g_{m,n} \text{ is borelian}\}. \tag{5.35}$$

Evidently, $\mathcal{N} \subset \mathcal{M}$. By a classical result from measure theory (see Berberian [24, page 178]) and the Lebesgue dominated convergence theorem, we deduce that $\mathcal{M} \subset \mathcal{N}$. Consequently, $\mathcal{M} = \mathcal{N}$. □

Proof of Theorem 5.3 continued. Let $v \in L^\infty(\Omega)$. There exist the sequences $\lambda_i \searrow 0$ and $h_i \to 0$ in $L^{p+1}(\Omega)$ such that

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_\Omega \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)\,ds\,dx. \tag{5.36}$$

We may assume that $h_i \to 0$ a.e. So

$$\begin{aligned}
\psi^0(u,v) &= \lim_{i\to\infty} \frac{1}{\lambda_i} \int_{[v>0]} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)\,ds\,dx \\
&\leq \int_{[v>0]} \left(\limsup_{i\to\infty} \frac{1}{\lambda_i} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)\,ds \right) dx \\
&\leq \int_{[v>0]} \overline{g}(x,u(x))\,v(x)\,dx.
\end{aligned} \tag{5.37}$$

So, for every $v \in L^\infty(\Omega)$,

$$\psi^0(u,v) \leq \int_{[v>0]} \overline{g}(x,u(x))\,v(x)\,dx. \tag{5.38}$$

Let us now assume that (5.30) is not true. So, there exist $\varepsilon > 0$, a set E with $|E| > 0$, and $w \in \partial\psi(u)$ such that for every $x \in E$,

$$w(x) \geq \overline{g}(x,u(x)) + \varepsilon. \tag{5.39}$$

Putting $v = \chi_E$ in (5.38) it follows that

$$\langle w, v\rangle = \int_E w \leq \psi^0(u,v) \leq \int_E \overline{g}(x,u(x))\,dx, \tag{5.40}$$

which contradicts (5.39). □

We assume in what follows that

$$g(x,0) = 0, \qquad \lim_{\varepsilon\searrow 0} \operatorname{ess\,sup} \left\{ \left| \frac{g(x,t)}{t} \right|;\ (x,t) \in \Omega \times [-\varepsilon,\varepsilon] \right\} < \lambda_1, \tag{5.41}$$

where λ_1 is the first eigenvalue of the operator $(-\Delta)$ in $H_0^1(\Omega)$. Furthermore, we assume that the following “technical” condition is fulfilled: there exist $\mu > 2$ and $r \geq 0$ such that

$$\mu G(x,t) \leq \begin{cases} t\underline{g}(x,t), & \text{a.e. } x \in \Omega,\ t \geq r \\ t\overline{g}(x,t), & \text{a.e. } x \in \Omega,\ t \leq -r \end{cases} \qquad \underline{g}(x,t) \geq 1 \text{ a.e. } x \in \Omega,\ t \geq r. \tag{5.42}$$

It follows from (5.1) and (5.41) that there exist some constants $0 < C_1 < \lambda_1$ and $C_2 > 0$ such that

$$|g(x,t)| \leq C_1|t| + C_2|t|^p, \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}. \tag{5.43}$$

Theorem 5.5. *Let $\alpha = \lambda_1/(\lambda_1 - C_1) > 1$ and let $a \in L^\infty(\Omega)$ be such that the operator $-\Delta + \alpha a$ is coercive. Assume, further, that the conditions* (5.1), (5.41), *and* (5.42) *are fulfilled.*

Then the multivalued elliptic problem

$$-\Delta u(x) + a(x)u(x) \in \left[\underline{g}(x,u(x)), \overline{g}(x,u(x))\right], \quad \text{a.e. } x \in \Omega \tag{5.44}$$

has a solution in $H_0^1(\Omega) \cap W^{2,q}(\Omega) \setminus \{0\}$, where q is the conjugated exponent of $p+1$.

Remark 5.6. The technical condition imposed to a is automatically fulfilled if $a \geq 0$ or, more generally, if $\|a^-\|_{L^\infty(\Omega)} < \lambda_1 \alpha^{-1}$.

Proof. Consider in the space $H_0^1(\Omega)$ the locally Lipschitz functional

$$\varphi(u) = \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega a(x)u^2(x)\,dx - \psi(u). \tag{5.45}$$

In order to prove the theorem, it is enough to show that φ has a critical point $u_0 \in H_0^1(\Omega)$ corresponding to a positive critical value. Indeed, we first observe that

$$\partial\varphi(u) = -\Delta u + a(x)u - \partial\psi_{|H_0^1(\Omega)}(u), \quad \text{in } H^{-1}(\Omega). \tag{5.46}$$

If u_0 is a critical point of φ, it follows that there exists $w \in \partial\psi_{|H_0^1(\Omega)}(u_0)$ such that

$$-\Delta u_0 + a(x)u_0 = w, \quad \text{in } H^{-1}(\Omega). \tag{5.47}$$

But $w \in L^q(\Omega)$. By a standard regularity result for elliptic equations, we deduce that $u_0 \in W^{2,q}(\Omega)$ and that u_0 is a solution of (5.44).

In order to prove that φ has such a critical point we apply Corollary 3.14. More precisely, we prove that φ satisfies the Palais-Smale condition, as well as the following “geometric” hypotheses:

$$\varphi(0) = 0 \text{ and there exists } v \in H_0^1(\Omega) \setminus \{0\} \text{ such that } \varphi(v) \leq 0; \tag{5.48}$$

$$\text{there exists } c > 0 \text{ and } 0 < R < \|v\| \text{ such that } \varphi \geq c \text{ on } \partial B(0,R). \tag{5.49}$$

Verification of (5.48). Evidently, $\varphi(0) = 0$. For the other assertion appearing in (5.48) we need

Lemma 5.7. *There exist positive constants C_1 and C_2 such that*

$$g(x,t) \geq C_1 t^{\mu-1} - C_2, \quad a.e.\ (x,t) \in \Omega \times [r,\infty). \tag{5.50}$$

Proof of Lemma 5.7. We show that

$$\underline{g} \leq g \leq \overline{g}, \quad \text{a.e. in } \Omega \times [r,\infty). \tag{5.51}$$

Assume for the moment that the relation (5.51) was proved. Then, by (5.42),

$$\mu \underline{G}(x,t) \leq t\underline{g}(x,t), \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}, \tag{5.52}$$

where

$$\underline{G}(x,t) = \int_0^t \underline{g}(x,s)ds. \tag{5.53}$$

Consequently, it is sufficient to prove (5.50) for $\underline{g}$ instead of g. □

Since $\underline{g}(x,\cdot) \in L^\infty_{\text{loc}}(\mathbb{R})$, it follows that $\underline{G}(x,\cdot) \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, provided $C > 0$ is sufficiently large so that

$$\mu \underline{G}(x,t) \leq C + t\underline{g}(x,t), \quad \text{a.e. } (x,t) \in \Omega \times [0,\infty). \tag{5.54}$$

We observe that (5.54) shows that the mapping

$$(0,+\infty) \ni t \longmapsto \frac{\underline{G}(x,t)}{t^\mu} - \frac{C}{\mu t^\mu} \tag{5.55}$$

is increasing. So, there exist R large enough and positive constants K_1, K_2 such that

$$\underline{G}(x,t) \geq K_1 t^\mu - K^2, \quad \text{a.e. } (x,t) \in \Omega \times [R,+\infty). \tag{5.56}$$

Relation (5.50) follows now from the above inequality and from (5.52).

For proving the second inequality appearing in (5.51), we observe that

$$\overline{g} = \lim_{n\to\infty} g_n \quad \text{in } L^\infty_{\text{loc}}(\Omega), \tag{5.57}$$

where

$$g_n(x,t) = \operatorname{ess\,sup}\left\{g(x,s);\ |t-s| \leq \frac{1}{n}\right\}. \tag{5.58}$$

For fixed $x \in \Omega$, it is sufficient to show that for every interval $I = [a,b] \subset \mathbb{R}$ we have

$$\int_I \overline{g}(x,t)dt \geq \int_I g(x,t)dt \tag{5.59}$$

and to use then a standard argument from measure theory.

Taking into account (5.57), it is enough to show that

$$\liminf_{n\to\infty}\int_I g_n(x,t)dt \geq \int_I g(x,t)dt. \tag{5.60}$$

We have

$$\begin{aligned}
\int_I g_n(x,t)dt &= \int_I \operatorname{ess\,sup}\left\{g(x,s);\ s\in\left[t-\frac{1}{n},t+\frac{1}{n}\right]\right\}\\
&\geq \int_I \frac{n}{2}\int_{t-1/n}^{t+1/n} g(x,s)ds = (\text{Fubini})\int_{a-1/n}^{b+1/n}\frac{n}{2}\int_{t_1(s)}^{t_2(s)} g(x,s)dt\,ds\\
&= \int_{a-1/n}^{b+1/n}\frac{n}{2}\left(t_2(s)-t_1(s)\right)g(x,s)ds\\
&= \int_a^b g(x,s)ds + \frac{n}{2}\int_{a-1/n}^{b+1/n}\left(s-\frac{1}{n}-a\right)g(x,s)ds\\
&\quad + \frac{n}{2}\int_{b-1/n}^{b+1/n}\left(b-s-\frac{1}{n}\right)g(x,s)ds \longrightarrow \int_a^b g(x,s)ds.
\end{aligned} \tag{5.61}$$

We have chosen n such that $2/n \leq b-a$, and

$$\begin{aligned}
t_1(s) &= \begin{cases} a & \text{if } a-\dfrac{1}{n}\leq s\leq a+\dfrac{1}{n},\\ s-\dfrac{1}{n} & \text{if } a+\dfrac{1}{n}\leq s\leq b+\dfrac{1}{n},\end{cases}\\
t_2(s) &= \begin{cases} s+\dfrac{1}{n} & \text{if } a-\dfrac{1}{n}\leq s\leq b-\dfrac{1}{n},\\ b & \text{if } b-\dfrac{1}{n}\leq s\leq b+\dfrac{1}{n}.\end{cases}
\end{aligned} \tag{5.62}$$

This concludes our proof. □

Proof of Theorem 5.5 continued. If $e_1 > 0$ denotes the first eigenfunction of the operator $-\Delta$ in $H_0^1(\Omega)$, then, for t large enough,

$$\begin{aligned}
\varphi(te_1) &\leq \frac{\lambda_1 t^2}{2}\|e_1\|^2_{L^2(\Omega)} + \frac{t^2}{2}\int_\Omega ae_1^2 - \psi(te_1)\\
&\leq \left(\frac{\lambda_1}{2}\|e_1\|^2_{L^2(\Omega)} + \int_\Omega ae_1^2\right)t^2 + C_2 t\int_\Omega e_1 - C_1' t^\mu\int_\Omega e_1^\mu < 0.
\end{aligned} \tag{5.63}$$

So, in order to obtain (5.48), it is enough to choose $v = te_1$, for t found above.

Verification of (5.49). Applying Poincaré's inequality, the Sobolev embedding theorem and (5.43), we deduce that for every $u\in H_0^1(\Omega)$,

$$\psi(u) \leq \frac{C_1}{2}\int_\Omega u^2 + \frac{C_2}{p+1}\int_\Omega |u|^{p+1} \leq \frac{C_1}{2\lambda_1}\|\nabla u\|^2_{L^2(\Omega)} + C'\|\nabla u\|^{p+1}_{L^2(\Omega)}. \tag{5.64}$$

Hence

$$\begin{aligned}\varphi(u) &\geq \frac{1}{2}\left(1-\frac{C_1}{\lambda_1}\right)\|\nabla u\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega au^2-C'\|\nabla u\|^{p+1}_{L^2(\Omega)}\\ &\geq \frac{1}{2}\left(1-\frac{C_1}{\lambda_1}-\frac{1}{\alpha+\varepsilon}\right)\|\nabla u\|^2_{L^2(\Omega)}-C'\|\nabla u\|^{p+1}_{L^2(\Omega)}\geq C>0,\end{aligned} \tag{5.65}$$

for $\varepsilon > 0$ sufficiently small, if $\|\nabla u\|_{L^2(\Omega)} = R$ is close to 0.

Verification of the Palais-Smale condition. Let (u_k) be a sequence in $H_0^1(\Omega)$ such that

$$\begin{aligned}&\varphi(u_k) \quad \text{is bounded,}\\ &\lim_{k\to\infty}\lambda(u_k)=0.\end{aligned} \tag{5.66}$$

The definition of λ and (5.28) imply the existence of a sequence (w_k) such that

$$\begin{aligned}&w_k\in\partial\psi_{|H_0^1(\Omega)}\subset L^q(\Omega),\\ &-\Delta u_k+a(x)u_k-w_k\longrightarrow 0 \quad \text{in } H^{-1}(\Omega).\end{aligned} \tag{5.67}$$

Since, by (5.1), the mapping G is locally bounded with respect to the variable t and uniformly with respect to x, the hypothesis (5.42) yields

$$\mu G(x,u(x)) \leq \begin{cases} u(x)\underline{g}(x,u(x))+C, & \text{a.e. in } [u\geq 0],\\ u(x)\overline{g}(x,u(x))+C, & \text{a.e. in } [u\leq 0],\end{cases} \tag{5.68}$$

where u is a measurable function defined on Ω, while C is a positive constant not depending on u. It follows that, for every $u \in H_0^1(\Omega)$,

$$\begin{aligned}\psi(u)&=\int_{[u\geq 0]}G(x,u(x))\,dx+\int_{[u\leq 0]}G(x,u(x))\,dx\\ &\leq\frac{1}{\mu}\int_{[u\geq 0]}u(x)\underline{g}(x,u(x))\,dx+\frac{1}{\mu}\int_{[u\leq 0]}u(x)\overline{g}(x,u(x))\,dx+C|\Omega|.\end{aligned} \tag{5.69}$$

This inequality and (5.30) show that, for every $u \in H_0^1(\Omega)$ and $w \in \partial\psi(u)$,

$$\psi(u)\leq\frac{1}{\mu}\int_\Omega u(x)w(x)dx+C'. \tag{5.70}$$

We first prove that the sequence (u_k) contains a subsequence which is weakly convergent in $H_0^1(\Omega)$. Indeed,

$$\begin{aligned}
\varphi(u_k) &= \frac{1}{2}\int_\Omega |\nabla u_k|^2 + \frac{1}{2}\int_\Omega a u_k^2 - \psi(u_k) \\
&= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega (|\nabla u_k|^2 + a u_k^2) + \frac{1}{\mu}\langle -\Delta u_k + a u_k - w_k, u_k\rangle + \frac{1}{\mu}\langle w_k, u_k\rangle - \psi(u_k) \\
&\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega (|\nabla u_k|^2 + a u_k^2) + \frac{1}{\mu}\langle -\Delta u_k + a u_k - w_k, u_k\rangle - C' \\
&\geq C''\int_\Omega |\nabla u_k|^2 - \frac{1}{\mu} o(1)\sqrt{\int_\Omega |\nabla u_k|^2} - C'.
\end{aligned} \tag{5.71}$$

This implies that the sequence (u_k) is bounded in $H_0^1(\Omega)$. So, up to a subsequence, (u_k) is weakly convergent to $u \in H_0^1(\Omega)$. Since the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is compact it follows that, up to a subsequence, (u_k) is strongly convergent in $L^{p+1}(\Omega)$. We remark that (u_k) is bounded in $L^q(\Omega)$. Since ψ is Lipschitz on the bounded subsets of $L^{p+1}(\Omega)$, it follows that (w_k) is bounded in $L^q(\Omega)$. From

$$\begin{aligned}
\|\nabla u_k\|^2_{L^2(\Omega)} = &\int_\Omega \nabla u_k \nabla u - \int_\Omega a u_k(u_k - u) \\
&+ \int_\Omega w_k(u_k - u) + \langle -\Delta u_k + a u_k - w_k, u_k - u\rangle_{H^{-1},H_0^1(\Omega)},
\end{aligned} \tag{5.72}$$

it follows that

$$\|\nabla u_k\|_{L^2(\Omega)} \longrightarrow \|\nabla u\|_{L^2(\Omega)}. \tag{5.73}$$

Since $H_0^1(\Omega)$ is a Hilbert space and

$$u_k \rightharpoonup u, \quad \|u_k\|_{H_0^1(\Omega)} \longrightarrow \|u\|_{H_0^1(\Omega)}, \tag{5.74}$$

we deduce that (u_k) converges to u in $H_0^1(\Omega)$. □

The method developed in the above proof shows that the result remains valid if the problem is affected by a perturbation which is *small* with respect to a certain topology. More precisely, the following stronger variant of Theorem 5.5 is true.

Theorem 5.8. *Under the same hypotheses as in Theorem 5.5, let $b \in L^q(\Omega)$ such that there exists $\delta > 0$ so that*

$$\|b\|_{L^\infty(\Omega)} < \delta. \tag{5.75}$$

Then the problem

$$-\Delta u(x) + a(x)u(x) + b(x) \in [\underline{g}(x,u(x)), \overline{g}(x,u(x))], \quad \text{a.e. } x \in \Omega \tag{5.76}$$

has a solution.

Proof. Define

$$\varphi(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega au^2 + \int_\Omega bu - \int_\Omega \int_0^{u(x)} g(x,t)dt\,dx. \tag{5.77}$$

We have seen that if $b = 0$, then the problem (5.76) has a solution. For $\|b\|_{L^\infty(\Omega)}$ sufficiently small, the verification of the Palais-Smale condition and of the geometric hypotheses (5.48) and (5.81) follows the same lines as in the proof of Theorem 5.5. □

5.3. The multivalued forced pendulum problem

Mawhin and Willem have studied in [143] a variational problem for periodic solutions of the forced pendulum equation. Their framework is related to a v-periodic energy functional φ defined on the Hilbert space H^1 of absolutely continuous functions $x : [0,1] \to \mathbb{R}$, where $v \in H^1 \setminus \{0\}$ is a constant function. They show that φ satisfies an appropriate Palais-Smale condition and that φ has a strict local minimum (actually an absolute minimum) at the origin and hence at each point kv, where k is an arbitrary integer. A direct consequence of the mountain pass theorem is that φ must have another critical point $x_0 \in H^1 \setminus \{kv;\ k \in \mathbb{Z}\}$. This critical point (which is a solution of the forced pendulum problem) is not a local minimum and satisfies $\varphi(x_0) > \varphi(0)$. Our purpose in this section is to extend the results of J. Mawhin and M. Willem in a multivalued formulation.

Consider the multivalued pendulum problem

$$\begin{gathered} x'' + f \in [\underline{g}(x), \overline{g}(x)], \\ x(0) = x(1), \end{gathered} \tag{5.78}$$

where

$$f \in L^p(0,1), \quad \text{for some } p > 1, \tag{5.79}$$

$$g \in L^\infty(\mathbb{R}) \text{ and there exists } T > 0 \text{ such that } g(x+T) = g(x), \quad \text{a.e. } x \in \mathbb{R}, \tag{5.80}$$

$$\int_0^T g(t)dt = \int_0^1 f(t)dt = 0. \tag{5.81}$$

The main result of this section is the following existence property for the multivalued forced pendulum problem.

Theorem 5.9. *If f and g are as above then problem* (5.78) *has at least two solutions in the space*

$$X := H^1_p(0,1) = \{x \in H^1(0,1);\ x(0) = x(1)\}. \tag{5.82}$$

Moreover, these solutions are distinct, in the sense that their difference is not an integer multiple of T.

Proof. As in the proof of Theorem 5.5, the critical points of the functional $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(x) = -\frac{1}{2}\int_0^1 x'^2 + \int_0^1 fx - \int_0^1 G(x) \tag{5.83}$$

are solutions of the problem (5.78). The details of the proof are, essentially, the same as above.

Since $\varphi(x+T) = \varphi(x)$, we may apply Theorem 4.15. All we have to do is to prove that φ verifies the condition $(\mathrm{PS})_{Z,c}$, for any c.

In order to do this, let (x_n) be a sequence in X such that

$$\lim_{n\to\infty} \varphi(x_n) = c, \tag{5.84}$$

$$\lim_{n\to\infty} \lambda(x_n) = 0. \tag{5.85}$$

Let

$$w_n \in \partial\varphi(x_n) \subset L^\infty(0,1) \tag{5.86}$$

be such that

$$\lambda(x_n) = x_n'' + f - w_n \longrightarrow 0 \quad \text{in } H^{-1}(0,1). \tag{5.87}$$

Observe that the last inclusion appearing in (5.86) is justified by the fact that

$$\underline{g} \circ x_n \le w_n \le \overline{g} \circ x_n, \tag{5.88}$$

and $\underline{g}, \overline{g} \in L^\infty(\mathbb{R})$.

By (5.85), after multiplication with x_n it follows that

$$\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 w_n x_n = o(1)\|x_n\|_{H_p^1}. \tag{5.89}$$

Then, by (5.84),

$$-\frac{1}{2}\int_0^1 (x_n')^2 + \int_0^1 fx_n - \int_0^1 G(x_n) \longrightarrow c. \tag{5.90}$$

So, there exist positive constants C_1 and C_2 such that

$$\int_0^1 (x_n')^2 \le C_1 + C_2\|x_n\|_{H_p^1}. \tag{5.91}$$

We observe that G is also T-periodic, so it is bounded.

For every n, replacing x_n with $x_n + k_nT$ for a convenable integer k_n, we can assume that

$$x_n(0) \in [0,T]. \tag{5.92}$$

We have obtained that the sequence (x_n) is bounded in $H_p^1(0,1)$.

Let $x \in H_p^1(0,1)$ be such that, up to a subsequence,

$$x_n \rightharpoonup x, \qquad x_n(0) \longrightarrow x(0). \tag{5.93}$$

Thus

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x \rangle + \int_0^1 w_n (x_n - x) - \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \longrightarrow \int_0^1 x'^2, \tag{5.94}$$

since $x_n \to x$ in $L^{p'}$, where p' is the conjugated exponent of p.

It follows that $x_n \to x$ in H_p^1, so the Palais-Smale condition $(\mathrm{PS})_{Z,c}$ is fulfilled. □

5.4. Multivalued Landesman-Lazer problems with resonance

In [131], Landesman and Lazer studied for the first time problems with resonance and they found sufficient conditions for the existence of a solution. We first recall the main definitions, in the framework of the singlevalued problems.

Let Ω be an open bounded open set in $\mathbb{R}^N$, and let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map. We consider the problem

$$-\Delta u = f(u), \quad \text{in } \Omega \; u \in H_0^1(\Omega). \tag{5.95}$$

To obtain information on the existence of solutions, as well as estimates on the number of solutions, it is necessary to have further information on f. In fact, the solutions of (5.95) depend in an essential manner on the asymptotic behaviour of f. Let us assume, for example, that f is asymptotic linear, that is $f(t)/t$ has a finite limit as $|t| \to \infty$. Let

$$a = \lim_{|t| \to \infty} \frac{f(t)}{t}. \tag{5.96}$$

We write

$$f(t) = at - g(t), \tag{5.97}$$

where

$$\lim_{|t| \to \infty} \frac{g(t)}{t} = 0. \tag{5.98}$$

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be the eigenvalues of the operator $(-\Delta)$ in $H_0^1(\Omega)$. The problem (5.95) is said to be with *resonance at infinity* if the number a defined in relation (5.96) is one of the eigenvalues λ_k of the linear operator $(-\Delta)$. With respect to the growth of g at infinity there are several "degrees" of resonance. If g has a "smaller" rate of increase at infinity, then its resonance is "stronger."

There are several situations:

(a) $\lim_{t \to \pm\infty} g(t) = \ell_\pm \in \mathbb{R}$ and $(\ell_+, \ell_-) \neq (0,0)$;
(b) $\lim_{|t| \to \infty} g(t) = 0$ and $\lim_{|t| \to \infty} \int_0^t g(s)ds = \pm\infty$;
(c) $\lim_{|t| \to \infty} g(t) = 0$ and $\lim_{|t| \to \infty} \int_0^t g(s)ds = \beta \in \mathbb{R}$.

The case (c) is called as the case of a *strong resonance*.

For a treatment of these cases, we refer to [4, 12, 13, 21, 53, 103].

5.4.1. Inequality problems with strong resonance

Consider the following multivalued variant of the Landesman-Lazer problem.

$$-\Delta u(x) - \lambda_1 u(x) \in [\underline{f}(u(x)), \overline{f}(u(x)) \quad \text{a.e. } x \in \Omega, \qquad u \in H_0^1(\Omega) \setminus \{0\}, \tag{5.99}$$

where

(i) $\Omega \subset \mathbb{R}^N$ is an open bounded set with sufficiently smooth boundary;
(ii) λ_1 (resp., e_1) is the first eigenvalue (resp., eigenfunction) of the operator $(-\Delta)$ in $H_0^1(\Omega)$;
(iii) $f \in L^\infty(\mathbb{R})$;
(iv) $\underline{f}(t) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{f(s);\ |t-s| < \varepsilon\}$, $\overline{f}(t) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup}\{f(s);\ |t-s| < \varepsilon\}$.

Consider in $H_0^1(\Omega)$ the functional $\varphi(u) = \varphi_1(u) - \varphi_2(u)$, where

$$\varphi_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 u^2), \qquad \varphi_2(u) = \int_\Omega F(u). \tag{5.100}$$

Observe first that φ is locally Lipschitz in $H_0^1(\Omega)$. Indeed, it is enough to show that φ_2 is locally Lipschitz in $H_0^1(\Omega)$, which follows from

$$\begin{aligned} |\varphi_1(u) - \varphi_2(u)| &= \left| \int_\Omega \left(\int_{u(x)}^{v(x)} f(t)dt \right) dx \right| \\ &\le \|f\|_{L^\infty} \cdot \|u - v\|_{L^1} \le C_1 \|u - v\|_{L^2} \le C_2 \|u - v\|_{H_0^1}. \end{aligned} \tag{5.101}$$

We will study problem (5.99) under the hypothesis

$$f(+\infty) := \lim_{t\to+\infty} f(t) = 0, \qquad F(+\infty) := \lim_{t\to+\infty} F(t) = 0. \tag{5.102}$$

Thus, by [131, 21], problem (5.99) becomes a Landesman-Lazer-type problem, with strong resonance at $+\infty$.

As an application of Corollary 3.17 we will prove the following sufficient condition for the existence of a solution to our problem.

Theorem 5.10. *Assume that f satisfies the condition* (5.99), *as well as*

$$-\infty \le F(-\infty) \le 0. \tag{5.103}$$

If $F(-\infty)$ is finite, we assume further that

$$\text{there exists } \eta > 0 \text{ such that } F \text{ is nonnegative on } (0,\eta) \text{ or } (-\eta, 0). \tag{5.104}$$

Under these hypotheses, problem (5.99) *has at least one solution.*

We will make use in the proof of the following auxiliary results.

Lemma 5.11. *Assume that* $f \in L^\infty(\mathbb{R})$ *and there exists* $F(\pm\infty) \in \overline{\mathbb{R}}$*. Moreover, assume that*

(i) $f(+\infty) = 0$ *if* $F(+\infty)$ *is finite,*
(ii) $f(-\infty) = 0$ *if* $F(-\infty)$ *is finite.*

Under these hypotheses, one has

$$\mathbb{R} \subset \{\alpha|\Omega|;\ \alpha = -F(\pm\infty)\} \subset \{c \in \mathbb{R};\ \varphi \text{ satisfies } (\mathrm{PS})_c\}. \tag{5.105}$$

Lemma 5.12. *Assume that* f *satisfies condition* (5.102)*. Then* φ *satisfies the Palais-Smale condition* $(PS)_c$*, for every* $c \neq 0$ *such that* $c < -F(-\infty)\cdot|\Omega|$.

Assume, for the moment, that these results have been proved.

Proof of Theorem 5.10. There are two distinct situations.

Case 1 ($F(-\infty)$). Is finite, that is, $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below, because

$$\varphi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) - \int_\Omega F(u) \geq -\int_\Omega F(u) \tag{5.106}$$

and, in virtue of the hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(\Omega)} \int_\Omega F(u) < +\infty. \tag{5.107}$$

Hence

$$-\infty < a := \inf_{H_0^1(\Omega)} \varphi \leq 0 = \varphi(0). \tag{5.108}$$

There exists a real number c, sufficiently small in absolute value and such that $F(ce_1) < 0$ (we observe that c may be chosen positive if $F > 0$ in $(0,\eta)$ and negative, if $F < 0$ in $(-\eta, 0)$). So, $\varphi(ce_1) < 0$, that is, $a < 0$. By Lemma 5.12, it follows that φ satisfies the Palais-Smale condition $(\mathrm{PS})_a$.

Case 2 ($F(-\infty) = -\infty$). Then, by Lemma 5.11, φ satisfies the condition $(\mathrm{PS})_c$, for every $c \neq 0$.

Let V be the orthogonal complement with respect to $H_0^1(\Omega)$ of the space spanned by e_1, that is,

$$H_0^1(\Omega) = \mathrm{Sp}\,\{e_1\} \oplus V. \tag{5.109}$$

For some fixed $t_0 > 0$, denote

$$\begin{gathered} V_0 := \{t_0 e_1 + v;\ v \in V\}, \\ a_0 := \inf_{V_0} \varphi. \end{gathered} \tag{5.110}$$

We remark that φ is coercive on V. Indeed, for every $v \in V$,

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda_1 v^2) - \int_\Omega F(v)\\ &\geq \frac{\lambda_2 - \lambda_1}{2}\|v\|^2_{H_0^1} - \int_\Omega F(v) \longrightarrow +\infty \quad \text{as } \|v\|_{H_0^1} \longrightarrow +\infty,\end{aligned} \tag{5.111}$$

because the first term in the right-hand side of (5.111) has a quadratic growth at infinity (t_0 being fixed), while $\int_\Omega F(v)$ is uniformly bounded (with respect to v), by the behaviour of F near $\pm\infty$. So, a_0 is attained, because of the coercivity of φ in V. Taking into account the boundedness of φ in $H_0^1(\Omega)$, it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

At this stage, there are again two possibilities.

(i) $a < 0$. Thus, by Lemma 5.12, φ satisfies $(\mathrm{PS})_a$. So, $a < 0$ is a critical value of φ.

(ii) $a = 0 \leq a_0$. If $a_0 = 0$ then, by a preceding remark, a_0 is achieved. So, there exists $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v). \tag{5.112}$$

Hence $u = t_0 e_1 + v \in H_0^1(\Omega) \setminus \{0\}$ is a critical point of φ, that is, a solution of the problem (5.99).

If $a_0 > 0$, we observe that φ satisfies $(\mathrm{PS})_b$ for every $b \neq 0$. Since $\lim_{t\to+\infty} \varphi(te_1) = 0$, we can apply Corollary 3.17 for $X = H_0^1(\Omega)$, $X_1 = V$, $X_2 = \mathrm{Sp}\,\{e_1\}$, $f = \varphi$, $z = t_0 e_1$. Therefore, φ has a critical value $c \geq a_0 > 0$. □

Proof of Lemma 5.11. We will assume, without loss of generality, that $F(-\infty) \notin \mathbb{R}$ and $F(+\infty) \in \mathbb{R}$. In this case, if c is a critical value such that φ does not satisfy $(\mathrm{PS})_c$, then it is enough to prove that $c = -F(+\infty) \cdot |\Omega|$. Since φ does not satisfy the condition $(\mathrm{PS})_c$, there exist $t_n \in \mathbb{R}$ and $v_n \in V$ such that the sequence $(u_n) \subset H_0^1(\Omega)$, where $u_n = t_n e_1 + v_n$, has no convergent subsequence, while

$$\lim_{n\to\infty} \varphi(u_n) = c, \tag{5.113}$$

$$\lim_{n\to\infty} \lambda(u_n) = 0. \tag{5.114}$$

Step 1 (*the sequence* (v_n) *is bounded in* $H_0^1(\Omega)$). By (5.114) and

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u), \tag{5.115}$$

it follows that there exists $w_n \in \partial\varphi_2(u_n)$ such that

$$-\Delta u_n - \lambda_1 u_n - w_n \longrightarrow 0 \quad \text{in } H^{-1}(\Omega). \tag{5.116}$$

So

$$\begin{aligned}&\langle -\Delta u_n - \lambda_1 u_n - w_n, v_n\rangle\\ &\quad = \int_\Omega |\nabla v_n|^2 - \lambda_1 \int_\Omega v_n^2 - \int_\Omega g_n(t_n e_1 + v_n) = o(\|v_n\|_{H_0^1}),\end{aligned} \tag{5.117}$$

as $n \to \infty$, where $\underline{f} \leq g_n \leq \overline{f}$. Since f is bounded, it follows that

$$\|v_n\|_{H_0^1}^2 - \lambda_1 \|v_n\|_{L^2}^2 = O(\|v_n\|_{H_0^1}). \tag{5.118}$$

So, there exists $C > 0$ such that, for every $n \geq 1$, $\|v_n\|_{H_0^1} \leq C$. Now, since (u_n) has no convergent subsequence, it follows that the sequence (u_n) has no convergent subsequence, too.

Step 2 ($t_n \to +\infty$). Since $\|v_n\|_{H_0^1} \leq C$ and the sequence $(t_n e_1 + v_n)$ has no convergent subsequence, it follows that $|t_n| \to +\infty$.

On the other hand, by Lebourg's mean value theorem, there exist $\theta \in (0,1)$ and $x^* \in \partial F(te_1(x) + \theta v(x))$ such that

$$\begin{aligned}
\varphi_2(te_1 + v) - \varphi_2(te_1) &= \int_\Omega \langle x^*, v(x)\rangle\, dx \\
&\leq \int_\Omega F^0(te_1(x) + v(x), v(x))\, dx \\
&= \int_\Omega \limsup_{\substack{y \to te_1(x)+v(x) \\ \lambda \searrow 0}} \frac{F(y + \lambda v(x)) - F(y)}{\lambda} dx \\
&\leq \|f\|_{L^\infty} \cdot \int_\Omega |v(x)|\, dx = \|f\|_{L^\infty} \cdot \|v\|_{L^1} \\
&\leq C_1 \|v\|_{H_0^1}.
\end{aligned} \tag{5.119}$$

A similar computation for $\varphi_2(te_1) - \varphi_2(te_1 + v)$ together with the above inequality shows that for every $t \in \mathbb{R}$ and all $v \in V$,

$$|\varphi_2(te_1 + v) - \varphi_2(te_1)| \leq C_2 \|v\|_{H_0^1}. \tag{5.120}$$

So, taking into account the boundedness of (v_n) in $H_0^1(\Omega)$, we find

$$|\varphi_2(t_n e_1 + v_n) - \varphi_2(t_n e_1)| \leq C. \tag{5.121}$$

Therefore, since $F(-\infty) \notin \mathbb{R}$ and

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(t_n e_1 + v_n) \longrightarrow c, \tag{5.122}$$

it follows that $t_n \to +\infty$. In this argument we have also used the fact that $\varphi_1(v_n)$ is bounded.

Step 3 ($\|v_n\|_{H_0^1} \to 0$ *as* $n \to \infty$). By (5.102) and Step 2 it follows that

$$\lim_{n\to\infty} \int_\Omega f(t_n e_1 + v_n) v_n = 0. \tag{5.123}$$

Using now (5.114) and Step 1 we deduce that

$$\lim_{n\to\infty} \|v_n\|_{H_0^1} = 0. \tag{5.124}$$

Step 4. We prove that

$$\lim_{t\to+\infty} \varphi_2(te_1 + v) = F(+\infty) \cdot |\Omega|, \tag{5.125}$$

uniformly on the bounded subsets of V. Indeed, assume the contrary. So, there exist $r > 0$, $t_n \to +\infty$, $v_n \in V$ with $\|v_n\| \le r$, such that (5.126) is not fulfilled. Thus, up to a subsequence, there exist $v \in H_0^1(\Omega)$ and $h \in L^2(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{5.126}$$

$$v_n \longrightarrow v \quad \text{strongly in } L^2(\Omega), \tag{5.127}$$

$$v_n(x) \longrightarrow v(x) \quad \text{a.e. } x \in \Omega, \tag{5.128}$$

$$|v_n(x)| \le h(x) \quad \text{a.e. } x \in \Omega. \tag{5.129}$$

For any $n \ge 1$ we define

$$\begin{gathered} A_n = \{x \in \Omega;\ t_n e_1(x) + v_n(x) < 0\}, \\ h_n(x) = F(t_n e_1 + v_n)\chi_{A_n}, \end{gathered} \tag{5.130}$$

where χ_A represents the characteristic function of the set A. By (5.129) and the choice of t_n, it follows that $|A_n| \to 0$ if $n \to \infty$.

Using (5.128) we remark that

$$h_n(x) \longrightarrow 0 \quad \text{a.e. } x \in \Omega. \tag{5.131}$$

Therefore,

$$\begin{aligned} |h_n(x)| &= \chi_{A_n}(x) \cdot \left| \int_0^{t_n e_1(x)+v_n(x)} f(s)ds \right| \\ &\le \chi_{A_n}(x) \cdot \|f\|_{L^\infty} \cdot |t_n e_1(x) + v_n(x)| \le C|v_n(x)| \\ &\le Ch(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{5.132}$$

So, by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{A_n} F(t_n e_1 + v_n) = 0. \tag{5.133}$$

On the other hand,

$$\lim_{n\to\infty} \int_{\Omega\setminus A_n} F(t_n e_1 + v_n) = F(+\infty) \cdot |\Omega|. \tag{5.134}$$

So

$$\lim_{n\to\infty} \varphi_2(t_n e_1 + v_n) = \lim_{n\to\infty} \int_\Omega F(t_n e_1 + v_n) = F(+\infty) \cdot |\Omega|, \tag{5.135}$$

which contradicts our initial assumption.

Step 5. Taking into account the preceding step and the fact that $\varphi(te_1+v) = \varphi_1(v) - \varphi_2(te_1+v)$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\varphi(t_ne_1+v_n) &= \lim_{n\to\infty}\varphi_1(v_n) - \lim_{n\to\infty}\varphi_2(t_ne_1+v_n)\\ &= -F(+\infty)\cdot|\Omega|,\end{aligned} \tag{5.136}$$

that is, $c = -F(+\infty)\cdot|\Omega|$, which concludes our proof. □

Proof of Lemma 5.12. It is enough to show that for every $c \neq 0$ and $(u_n) \subset H_0^1(\Omega)$ such that

$$\begin{aligned}\varphi(u_n) &\longrightarrow c,\\ \lambda(u_n) &\longrightarrow 0,\\ \|u_n\| &\longrightarrow \infty,\end{aligned} \tag{5.137}$$

we have $c \geq -F(-\infty)\cdot|\Omega|$.

Let $t_n \in \mathbb{R}$ and $v_n \in V$ be such that for every $n \geq 1$,

$$u_n = t_ne_1 + v_n. \tag{5.138}$$

As we have already remarked,

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(u_n). \tag{5.139}$$

Moreover,

$$\varphi_1(v) = \frac{1}{2}\int_\Omega\left(|\nabla v|^2 - \lambda_1v^2\right) \geq \frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\cdot\|v\|_{H_0^1}^2 \longrightarrow +\infty \quad \text{as } \|v\|_{H_0^1} \longrightarrow \infty. \tag{5.140}$$

So, φ_1 is positive and coercive on V. We also have that φ_2 is bounded from below, by (5.102). So, again by (5.102), we conclude that the sequence (v_n) is bounded in $H_0^1(\Omega)$. Thus, there exists $v \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{aligned}v_n &\rightharpoonup v \quad \text{weakly in } H_0^1(\Omega),\\ v_n &\longrightarrow v \quad \text{strongly in } L^2(\Omega),\\ v_n(x) &\longrightarrow v(x) \quad \text{a.e. } x\in\Omega,\\ |v_n(x)| &\leq h(x) \quad \text{a.e. } x\in\Omega,\end{aligned} \tag{5.141}$$

for some $h \in L^2(\Omega)$.

Since $\|u_n\|_{H_0^1} \to \infty$ and (v_n) is bounded in $H_0^1(\Omega)$, it follows that $|t_n| \to +\infty$.

Assume for the moment that we have already proved that $\|v_n\|_{H_0^1} \to 0$, if $t_n \to +\infty$. So,

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(u_n) \longrightarrow 0 \quad \text{if } n \longrightarrow \infty. \tag{5.142}$$

Here, for proving that $\varphi_2(u_n) \to 0$, we have used our assumption (5.102). The last relation yields a contradiction, since $\varphi(u_n) \to c \neq 0$. So, $t_n \to -\infty$.

Moreover, since $\varphi(u) \geq -\varphi_2(u)$ and F is bounded from below, it follows that

$$\begin{aligned} c &= \liminf_{n\to\infty} \varphi(u_n) \geq \liminf_{n\to\infty} \left(-\varphi_2(u_n)\right) \\ &= -\limsup_{n\to\infty} \int_\Omega F(u_n) \geq -\int_\Omega \limsup_{n\to\infty} F(u_n) \\ &= -F(-\infty)\cdot|\Omega|, \end{aligned} \tag{5.143}$$

which gives the desired contradiction.

So, for concluding the proof, we have to show that

$$||v_n||_{H_0^1} \longrightarrow 0 \quad \text{if } t_n \longrightarrow +\infty. \tag{5.144}$$

Since

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u), \tag{5.145}$$

it follows from (5.137) that there exists $w_n \in \partial\varphi_2(u_n)$ such that

$$-\Delta u_n - \lambda_1 u_n - w_n \longrightarrow 0 \quad \text{in } H^{-1}(\Omega). \tag{5.146}$$

Thus

$$\begin{aligned} &\langle -\Delta u_n - \lambda_1 u_n - w_n, v_n\rangle \\ &\quad = \int_\Omega |\nabla v_n|^2 - \lambda_1 \int_\Omega v_n^2 - \int_\Omega g_n(t_n e_1 + v_n) v_n = o(||v_n||) \quad \text{if } n \to \infty, \end{aligned} \tag{5.147}$$

where $\underline{f} \leq g_n \leq \overline{f}$.

Now, for concluding the proof, it is sufficient to show that the last term tends to 0, as $n \to \infty$.

Let $\varepsilon > 0$. Because $f(+\infty) = 0$, it follows that there exists $T > 0$ such that

$$|f(t)| \leq \varepsilon \quad \text{a.e. } t \geq T. \tag{5.148}$$

Set

$$A_n = \{x \in \Omega;\ t_n e_1(x) + v_n(x) \geq T\}, \qquad B_n = \Omega \setminus A_n. \tag{5.149}$$

We remark that for every $x \in B_n$,

$$|t_n e_1(x) + v_n(x)| \leq |v_n(x)| + T. \tag{5.150}$$

So, for every $x \in B_n$,

$$|g_n(t_n e_1(x) + v_n(x)) v_n(x)| \cdot \chi_{B_n}(x) \leq \|f\|_{L^\infty} \cdot h(x). \tag{5.151}$$

By

$$\chi_{B_n}(x) \longrightarrow 0 \quad \text{a.e. } x \in \Omega \tag{5.152}$$

and the Lebesgue dominated convergence theorem it follows that

$$\int_{B_n} g_n(t_n e_1 + v_n)v_n \longrightarrow 0 \quad \text{if } n \longrightarrow \infty. \tag{5.153}$$

On the other hand, we observe that

$$\left| \int_{A_n} g_n(t_n e_1 + v_n)v_n \right| \le \varepsilon \int_{A_n} |v_n| \le \varepsilon \|h\|_{L^1}. \tag{5.154}$$

By (5.153) and (5.154) it follows that

$$\lim_{n\to\infty} \int_\Omega g_n(u_n)v_n = 0, \tag{5.155}$$

which concludes our proof. □

5.4.2. Inequality problems with mixed resonance

Let X be a Banach space. Assume that there exists $w \in X \setminus \{0\}$, which can be supposed to have the norm 1, and a linear subspace Y of X such that

$$X = \mathrm{Sp}\{w\} \oplus Y. \tag{5.156}$$

Definition 5.13. A subset A of X is called to be w-bounded if there exists $r \in \mathbb{R}$ such that

$$A \subset \{x = tw + y;\ t < r, y \in Y\}. \tag{5.157}$$

A functional $f : X \to \mathbb{R}$ is said to be w-coercive (or, coercive with respect to the decomposition (5.156)) if

$$\lim_{t\to\infty} f(tw + y) = +\infty, \tag{5.158}$$

uniformly with respect to $y \in Y$.

The functional f is called w-bounded from below if there exists $a \in \mathbb{R}$ such that the set $[f \le a]$ is w-bounded.

The following result is an extension of Proposition 3.34.

Proposition 5.14. *Let f be a w-bounded from below locally Lipschitz functional. Assume that there exists $c \in \mathbb{R}$ such that f satisfies the condition (s-PS)$_c$, and the set $[f \le a]$ is w-bounded from below for every $a < c$.*

Then there exists $\alpha > 0$ such that the set $[f \le c + \alpha]$ is w-bounded.

Proof. Assume the contrary. So, for every $\alpha > 0$, the set $[f \le c + \alpha]$ is not w-bounded. Thus, for every $n \ge 1$, there exists $r_n \ge n$ such that

$$\left[f \le c - \frac{1}{n^2}\right] \subset A_n := \{x = tw + y;\ t < r_n,\ y \in Y\}. \tag{5.159}$$

Therefore,

$$c_n := \inf_{X \setminus A_n} f \geq c - \frac{1}{n^2}. \tag{5.160}$$

Since the set $[f \leq c + 1/n^2]$ is not w-bounded, there exists a sequence (z_n) in X such that $z_n = t_n w + y_n$ and

$$t_n \geq r_n + 1 + \frac{1}{n}, \tag{5.161}$$

$$f(z_n) \leq c + \frac{1}{n^2}. \tag{5.162}$$

It follows that $z_n \in X \setminus A_n$ and, by (5.160) and (5.162), we find

$$f(z_n) \leq c_n + \frac{2}{n^2}. \tag{5.163}$$

We apply Ekeland's variational principle to f restricted to the set $X \setminus A_n$, provided $\varepsilon = 2/n^2$ and $\lambda = 1/n$. So, there exists $x_n = t_n' w + y_n' \in X \setminus A_n$ such that for every $x \in X \setminus A_n$,

$$c \leq f(x_n) \leq f(z_n), \tag{5.164}$$

$$f(x) \geq f(x_n) - \frac{2}{n} ||x - x_n||, \tag{5.165}$$

$$||x_n - z_n|| \leq \frac{1}{n}. \tag{5.166}$$

Let P ($\|P\| = 1$) be the projection of X on Sp $\{w\}$. Using the continuity of P and the relation (5.166) we obtain

$$|t_n - t_n'| \leq \frac{1}{n}. \tag{5.167}$$

So,

$$|t_n'| \geq r_n + 1. \tag{5.168}$$

Therefore, x_n is an interior point of the set A_n. By (5.165) it follows that for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{2}{n}\|v\|. \tag{5.169}$$

This relation and the fact that f satisfies the condition (s-PS)$_c$ imply that the sequence (x_n) contains a convergent subsequence, contradiction, because this sequence is not w-bounded. □

As an application of these results we will study the following multivalued Landesman-Lazer problem with mixed resonance.

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with the boundary sufficiently smooth. Consider the problem

$$-\Delta u(x) - \lambda_1 u(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \quad \text{a.e. } x \in \Omega \; u \in H_0^1(\Omega). \tag{5.170}$$

We will study this problem under the following hypotheses:

(f1) $f \in L^\infty(\mathbb{R})$;

(f2) if $F(t) = \int_0^1 f(s)ds$, then $\lim_{t\to+\infty} F(t) = 0$;

(f3) $\lim_{t\to-\infty} F(t) = +\infty$;

(f4) there exists $\alpha < 1/2(\lambda_2 - \lambda_1)$ such that, for every $t \in \mathbb{R}$, $F(t) \le \alpha t^2$.

Define on the space $H_0^1(\Omega)$ the functional $\varphi(u) = \varphi_1(u) - \varphi_2(u)$, where

$$\begin{aligned} \varphi_1(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right)dx, \\ \varphi_2(u) &= \int_\Omega F(u)dx. \end{aligned} \tag{5.171}$$

As in the preceding section, we observe that the functional φ is locally Lipschitz.

Let Y be the orthogonal complement of the space spanned by e_1, that is,

$$H_0^1(\Omega) = \operatorname{Sp}\{e_1\} \oplus Y. \tag{5.172}$$

Lemma 5.15. *The following properties hold true:*

(i) *there exists $r_0 > 0$ such that for every $t \in \mathbb{R}$ and $v \in Y$ with $\|v\|_{H_0^1} \ge r_0$,*

$$\varphi(te_1 + v) \ge -\varphi_2(te_1); \tag{5.173}$$

(ii)

$$\lim_{t\to+\infty} \varphi(te_1 + v) = \varphi_1(v), \tag{5.174}$$

uniformly on the bounded subsets of Y.

Proof. (i) We first observe that for every $t \in \mathbb{R}$ and all $v \in Y$,

$$\varphi_1(te_1 + v) = \varphi_1(v). \tag{5.175}$$

So,

$$\varphi(te_1 + v) = \varphi_1(v) - \varphi_2(te_1 + v). \tag{5.176}$$

On the other hand,

$$\begin{aligned} \varphi_2(te_1 + v) - \varphi_2(te_1) &= \int_\Omega \left(\int_{te_1(x)}^{te_1(x)+v(x)} f(s)ds\right)dx \\ &= \int_\Omega \left(F(te_1(x) + v(x)) - F(te_1(x))\right)dx. \end{aligned} \tag{5.177}$$

By the Lebourg mean value theorem, there exist $\theta \in (0,1)$ and $x^* \in \partial F(te_1(x) + \theta v(x))$ such that

$$F(te_1(x) + v(x)) - F(te_1(x)) = \langle x^*, v(x)\rangle. \tag{5.178}$$

The relation (5.177) becomes

$$\begin{aligned}\varphi_2(te_1+v)-\varphi_2(te_1) &= \int_\Omega \langle x^*, v(x)\rangle\,dx \\ &\le \int_\Omega F^0(te_1(x)+\theta v(x), v(x))\,dx \\ &= \int_\Omega \limsup_{\substack{y\to te_1(x)+\theta v(x)\\ \lambda\searrow 0}} \frac{F(y+\lambda v(x))-F(y)}{\lambda}\,dx \\ &\le \|f\|_{L^\infty}\cdot\int_\Omega |v(x)|\,dx \\ &= \|f\|_{L^\infty}\cdot\|v\|_{L^1} \le C\|v\|_{H_0^1}.\end{aligned} \tag{5.179}$$

By (5.176) and (5.179) it follows that

$$\varphi(te_1+v) \ge \varphi_1(v) - C\|v\|_{H_0^1} - \varphi_2(te_1). \tag{5.180}$$

So, for concluding the proof, it is enough to choose $r_0 > 0$ such that

$$\varphi_1(v) - C\|v\|_{H_0^1} \ge 0, \tag{5.181}$$

for every $v \in Y$ cu $\|v\|_{H_0^1} \ge r_0$. This choice is possible if we take into account the variational characterization of λ_2 and the fact that λ_1 is a simple eigenvalue. Indeed,

$$\begin{aligned}\varphi_1(v) - C\|v\|_{H_0^1} &= \frac{1}{2}\left(\|\nabla v\|_{L^2}^2 - \lambda_1\|v\|_{L^2}^2\right) - C\|\nabla v\|_{L^2} \\ &\ge \left(\frac{1}{2}-\varepsilon\right)\|\nabla v\|_{L^2}^2 - \frac{\lambda_1}{2}\|v\|_{L^2}^2 \\ &\ge \left(\frac{1}{2}-\varepsilon\right)\lambda_2\|v\|_{L^2}^2 - \frac{\lambda_1}{2}\|v\|_{L^2}^2 \ge 0,\end{aligned} \tag{5.182}$$

for $\varepsilon > 0$ sufficiently small and $\|v\|_{L^2}$ (hence, $\|v\|_{H_0^1}$) large enough.

(ii) Using relation (5.176), our statement is equivalent with

$$\lim_{t\to+\infty} \varphi_2(te_1+v) = 0, \tag{5.183}$$

uniformly with respect to v, in every closed ball.

Assume, by contradiction, that there exist $R > 0$, $t_n \to +\infty$ and $v_n \in Y$ with $\|v_n\| \le R$ such that (5.183) does not hold. So, up to a subsequence, we may assume that there exist $v \in H_0^1(\Omega)$ and $h \in L^1(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{5.184}$$

$$v_n \longrightarrow v \quad \text{strongly in } L^2(\Omega), \tag{5.185}$$

$$v_n(x) \longrightarrow v(x) \quad \text{for a.e. } x \in \Omega, \tag{5.186}$$

$$|v_n(x)| \le h(x) \quad \text{a.e. } x \in \Omega. \tag{5.187}$$

For every $n \geq 1$, denote

$$A_n = \{x \in \Omega;\ t_n e_1(x) + v_n(x) < 0\},$$
$$g_n = F(t_n e_1 + v_n)\chi_{A_n}. \tag{5.188}$$

By (5.187) and the choice of t_n it follows that $|A_n| \to 0$.
Using now (5.186), we observe that

$$g_n(x) \longrightarrow 0 \quad \text{a.e. } x \in \Omega. \tag{5.189}$$

By (f1) and (5.187) it follows that

$$\begin{aligned} |g_n(x)| &= \chi_{A_n}(x) \cdot \left| \int_0^{t_n e_1(x)+v_n(x)} f(s)ds \right| \\ &\leq \chi_{A_n}(x) \cdot \|f\|_{L^\infty} \cdot |t_n e_1(x) + v_n(x)| \\ &\leq C|v_n(x)| \leq Ch(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{5.190}$$

Thus, by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{A_n} F(t_n e_1 + v_n)dx = 0. \tag{5.191}$$

By (f2) it follows that F is bounded on $[0, \infty)$. Using again (f2) we find

$$\lim_{n\to\infty} \int_{\Omega \setminus A_n} F(t_n e_1 + v_n)dx = 0. \tag{5.192}$$

So

$$\lim_{n\to\infty} \varphi_2(t_n e_1 + v_n) = \lim_{n\to\infty} \int_\Omega F(t_n e_1 + v_n)dx = 0, \tag{5.193}$$

which contradicts our initial assumption. □

Remark 5.16. As a consequence of the above result,

$$\lim_{t\to+\infty} \varphi(te_1) = 0,$$
$$\liminf_{t\to+\infty} \inf_{v\in Y} \varphi(te_1 + v) \geq 0. \tag{5.194}$$

Thus, the set $[\varphi \leq a]$ is e_1-bounded if $a < 0$ and is not e_1-bounded for $a > 0$. Moreover, φ is not bounded from below, because

$$\lim_{t\to-\infty} \varphi(te_1) = -\lim_{t\to-\infty} \varphi_2(te_1) = -\lim_{t\to-\infty} \int_\Omega F(te_1) = -\infty. \tag{5.195}$$

Thus, by Proposition 5.14, it follows that φ does not satisfy the condition $(\text{s} - \text{PS})_0$.

Theorem 5.17. *Assume that f does not satisfy conditions (f1)–(f4). If the functional φ has the strong Palais-Smale property (s-PS)$_a$, for every $a \neq 0$, then the multivalued problem* (5.170) *has at least a nontrivial solution.*

Proof. It is sufficient to show that φ has a critical point $u_0 \in H_0^1(\Omega) \setminus \{0\}$. It is obvious that

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u) \quad \text{in } H^{-1}(\Omega). \tag{5.196}$$

If u_0 is a critical point of φ, then there exists $w \in \partial\varphi_2(u_0)$ such that

$$-\Delta u_0 - \lambda_1 u_0 = w \quad \text{in } H^{-1}(\Omega). \tag{5.197}$$

Set

$$Y_1 = \{e_1 + v;\ v \in Y\}. \tag{5.198}$$

We first remark the coercivity of φ on Y_1:

$$\begin{aligned}
\varphi(e_1 + v) &= \frac{1}{2}\left(\|\nabla\|_{L^2}^2 - \lambda_1\|v\|_{L^2}^2\right) - \lambda_1 \int_\Omega e_1 v dx - \int_\Omega G(e_1 + v)\,dx \\
&\geq \frac{1}{2}\left(\|\nabla v\|_{L^2}^2 - \lambda_1\|v\|_{L^2}^2\right) - \lambda_1\|v\|_{L^2} - \alpha\left(1 + \|v\|_{L^2}^2 + 2\int_\Omega e_1 v\,dx\right) \\
&\geq \left(\frac{\lambda_2 - \lambda_1}{2} - \alpha\right)\|v\|_{L^2}^2 - (\lambda_1 + 2|\alpha|)\|v\|_{L^2}^2 - \alpha,
\end{aligned} \tag{5.199}$$

which tends to $+\infty$ as $\|v\|_{L^2} \to \infty$, so, as $\|v\|_{H_0^1} \to \infty$.

Putting

$$m_1 = \inf_{Y_1} \varphi > -\infty, \tag{5.200}$$

it follows that m_1 is attained, by the coercivity of φ pe Y_1. Thus, there exists $u_0 = e_1 + v_0 \in Y_1$ such that $\varphi(u_0) = m_1$. There are two possibilities:

(1) $m_1 > 0$. It follows from (f1) and (f3) that

$$\begin{aligned}
\lim_{t\to+\infty} \varphi(te_1) &= -\lim_{t\to+\infty} \varphi_2(te_1) = 0, \\
\lim_{t\to-\infty} \varphi(te_1) &= -\lim_{t\to-\infty} \varphi_2(te_1) = -\infty.
\end{aligned} \tag{5.201}$$

Since $m_1 > 0 = \varphi(0)$ and φ has the property $(s - \mathrm{PS})_a$ for every $a > 0$, it follows from Corollary 3.17 that φ has a critical value $c \geq m_1 > 0$.

(2) $m_1 \leq 0$. Let

$$\begin{aligned}
W &= \{te_1 + v;\ t \geq 0,\ v \in Y\}, \\
c &= \inf_W \varphi.
\end{aligned} \tag{5.202}$$

So, $c \leq m_1 \leq 0$.

If $c < 0$, it follows by (s-PS)$_c$ and the Ekeland variational principle that c is a critical value of φ.

If $c = m_1 = 0$, then u_0 is a local minimum point of φ, because

$$\varphi(u_0) = \varphi(e_1 + v_0) = 0 \leq \varphi(te_1 + v), \tag{5.203}$$

for every $t \geq 0$ and $v \in Y$.

So, $u_0 \neq 0$ is a critical point of φ. □

5.5. Hartman-Stampacchia theory for hemivariational inequalities

The theorem of Hartman-Stampacchia asserts that if V is a finite dimensional Banach space, $K \subset V$ is compact and convex, $A : K \to V^\star$ is continuous, then there exists $u \in K$ such that for every $v \in K$,

$$\langle Au, v - u \rangle \geq 0. \tag{5.204}$$

We state below the following elementary proof which is due to H. Brezis. We first observe that since V is a finite dimensional space (say, $\dim V = n$), then we can identify $V^\star$ with $\mathbb{R}^n$ through the mapping $\pi : V^\star \to \mathbb{R}^n$ defined by

$$\langle a, x \rangle = (\pi a, x) \quad \forall a \in V^\star,\ x \in \mathbb{R}^n, \tag{5.205}$$

where $(\cdot, \cdot)$ stands for the scalar product in $\mathbb{R}^n$. This enables us to observe that for proving (5.204) it is enough to show that there is some $u \in K$ such that for every $v \in K$,

$$(u, v - u) \geq (u - \pi Au, v - u). \tag{5.206}$$

But the mapping $P_K(I - \pi A) : K \to K$ is continuous, where $I : K \to K$ is the identity map and $P_K : V \to K$ denotes the projection map onto the compact and convex set K. So, by the Brouwer fixed point theorem, there exists $u \in K$ such that $u = P_K(I - \pi A)u$. Next, using the characterization of the projection on a closed convex subset of a Hilbert space (see [30, Theorem V.2]), we deduce (5.206).

If we weak the hypotheses and consider the case where K is a closed and convex subset of the finite dimensional space V, Hartman and Stampacchia proved (see [121, Theorem I.4.2]) that a necessary and sufficient condition which ensures the existence of a solution to problem (5.204) is that there is some $R > 0$ such that a solution u of (1) with $\|u\| \leq R$ satisfies $\|u\| < R$.

Since the above elementary arguments are no longer available in the setting of *hemivariational inequalities*, we aim to extend in this section existence results of this type for more general inequality problems of this type. We point out that these problems appear as a natural generalization of variational inequalities but they are more general, in the sense that they are not equivalent to minimum problems but give rise to substationarity problems.

Let V be a real Banach space and let $T : V \to L^p(\Omega, \mathbb{R}^k)$ be a linear and continuous operator, where $1 \leq p < \infty$, $k \geq 1$, and Ω is a bounded open set in $\mathbb{R}^N$. Throughout this section we assume that K is a subset of V, $A : K \to V^*$ is a nonlinear operator and $j = j(x, y) : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable $y \in \mathbb{R}^k$ and satisfies the following assumption:

(j) there exists $h_1 \in L^{p/(p-1)}(\Omega, \mathbb{R})$ and $h_2 \in L^\infty(\Omega, \mathbb{R})$ such that

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1}, \tag{5.207}$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $z \in \partial j(x, y)$.

Denoting $Tu = \hat{u}$, $u \in V$, our aim in this section is to study the following inequality problem:

(P) find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0. \tag{5.208}$$

We have denoted by $j^0(x, y; h)$ the (partial) Clarke derivative of the locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $h \in \mathbb{R}^k$, where $x \in \Omega$, and by $\partial j(x, y)$ the Clarke generalized gradient of this mapping at $y \in \mathbb{R}^k$. The Euclidean norm in $\mathbb{R}^k$, $k \geq 1$, respectively, the duality pairing between a Banach space and its dual will be denoted by $|\cdot|$, respectively, $\langle \cdot, \cdot \rangle$. We also denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega, \mathbb{R}^k)$ defined by

$$\|\hat{u}\|_p = \left(\int_\Omega |\hat{u}(x)|^p\, dx \right)^{1/p}, \quad 1 \leq p < \infty. \tag{5.209}$$

A basic tool in our proofs is the following auxiliary result.

Lemma 5.18. (a) *If it is satisfied the assumption (j) and V_1, V_2 are nonempty subsets of V, then the mapping $V_1 \times V_2 \to \mathbb{R}$ defined by*

$$(u, v) \longrightarrow \int_\Omega j^0(x, \hat{u}(x), \hat{v}(x))\, dx \tag{5.210}$$

is upper semicontinuous;

(b) *moreover, if $T : V \to L^p(\Omega, \mathbb{R}^k)$ is a linear compact operator, then the above mapping is weakly upper semicontinuous.*

Proof. (a) Let $\{(u_m, v_m)\}_m \subset V_1 \times V_2$ be a sequence converging to $(u, v) \in V_1 \times V_2$, as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbb{R}^k)$ is continuous, it follows that

$$\hat{u}_m \longrightarrow \hat{u}, \quad \hat{v}_m \longrightarrow \hat{v} \quad \text{in } L^p(\Omega, \mathbb{R}^k), \text{ as } m \longrightarrow \infty \tag{5.211}$$

There exists a subsequence $\{(\hat{u}_n, \hat{v}_n)\}$ of the sequence $\{(\hat{u}_m, \hat{v}_m)\}$ such that

$$\limsup_{m \to \infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))\, dx = \lim_{n \to \infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))\, dx. \tag{5.212}$$

By Proposition 4.11 in Brezis [30], one may suppose the existence of two functions $\hat{u}_0$, $\hat{v}_0 \in L^p(\Omega, \mathbb{R}^+)$, and of two subsequences of $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ denoted again by the same symbols and such that

$$\begin{gathered} |\hat{u}_n(x)| \leq \hat{u}_0(x), \qquad |\hat{v}_n(x)| \leq \hat{v}_0(x), \\ \hat{u}_n(x) \longrightarrow \hat{u}(x), \quad \hat{v}_n(x) \longrightarrow \hat{v}(x), \quad \text{as } n \longrightarrow \infty \end{gathered} \tag{5.213}$$

for a.e. $x \in \Omega$. On the other hand, for each x where holds true the condition (j) and for each $y, h \in \mathbb{R}^k$, there exists $z \in \partial j(x, y)$ such that

$$j^0(x, y; h) = \langle z, h \rangle = \max\{\langle w, h \rangle : w \in \partial j(x, y)\}. \tag{5.214}$$

Now, by (j),

$$|j^0(x, y; h)| \leq |z||h| \leq (h_1(x) + h_2(x)|y|^{p-1}) \cdot |h|. \tag{5.215}$$

Consequently, denoting $F(x) = (h_1(x) + h_2(x)|\hat{u}_0(x)|^{p-1})|\hat{v}_0(x)|$, we find that

$$|j^0(x, \hat{u}_n(x); \hat{v}_n(x))| \leq F(x), \tag{5.216}$$

for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$.

From Hölder's inequality and from condition (j) for the functions h_1 and h_2, it follows that $F \in L^1(\Omega, \mathbb{R})$. Fatou's lemma yields

$$\lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))\,dx \leq \int_\Omega \limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x))\,dx. \tag{5.217}$$

Next, by the upper-semicontinuity of the mapping $j^0(x, \cdot; \cdot)$ we deduce that

$$\limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x)) \leq j^0(x, \hat{u}(x); \hat{v}(x)), \tag{5.218}$$

for a.e. $x \in \Omega$, because

$$\hat{u}_n(x) \longrightarrow \hat{u}(x), \qquad \hat{v}_n(x) \longrightarrow \hat{v}(x), \quad \text{as } n \longrightarrow \infty, \tag{5.219}$$

for a.e. $x \in \Omega$. Hence

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))\,dx \leq \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))\,dx, \tag{5.220}$$

which proves the upper-semicontinuity of the mapping defined by (5.210).

(b) Let $\{(u_m, v_m)\}_m \subset V_1 \times V_2$ be now a sequence weakly-converging to $\{u, v\} \in V_1 \times V_2$, as $m \to \infty$. Thus $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbb{R}^k)$ is a linear compact operator, it follows that

$$\hat{u}_m \longrightarrow \hat{u}, \quad \hat{v}_m \longrightarrow \hat{v} \quad \text{in } L^p(\Omega, \mathbb{R}^k). \tag{5.221}$$

From now on the proof follows the same lines as in case (a). □

5.5.1. Abstract existence results

We first recall the following definitions.

Definition 5.19. The operator $A : K \to V^*$ is w^*-demicontinuous if for any sequence $\{u_n\} \subset K$ converging to u, the sequence $\{Au_n\}$ converges to Au for the w^*-topology in V^*.

Definition 5.20. The operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K if for any finite dimensional space $F \subset V$, which intersects K, the operator $A|_{K\cap F}$ is demicontinuous, that is $\{Au_n\}$ converges weakly to Au in V^* for each sequence $\{u_n\} \subset K \cap F$ which converges to u.

Remark 5.21. In reflexive Banach spaces, the following hold:

(a) the w^*-demicontinuity and demicontinuity are the same;
(b) a demicontinuous operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K.

The following result is a generalized form of the Hartman-Stampacchia theorem for the case of hemivariational inequalities in finite dimensional real Banach spaces.

Theorem 5.22. *Let V be a finite dimensional Banach space and let K be a compact and convex subset of V. If assumption (j) is fulfilled and if $A : K \to V^*$ is a continuous operator, then problem (P) has at least a solution.*

Proof. Arguing by contradiction, for every $u \in K$, there is some $v = v_u \in K$ such that

$$\langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx < 0. \tag{5.222}$$

For every $v \in K$, set

$$N(v) = \left\{u \in K; \langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx < 0\right\}. \tag{5.223}$$

For any fixed $v \in K$ the mapping $K \to \mathbb{R}$ defined by convex hull

$$u \longmapsto \langle Au, v-u\rangle + \int_\Omega j^0(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx \tag{5.224}$$

is upper semicontinuous, by Lemma 5.18 and the continuity of A. Thus, by the definition of the upper semi-continuity, $N(v)$ is an open set. Our initial assumption implies that $\{N(v); v \in K\}$ is a covering of K. Hence, by the compactness of K, there exist $v_1,\dots,v_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} N(v_j). \tag{5.225}$$

Let $\rho_j(u)$ be the distance from u to $K \setminus N(v_j)$. Then ρ_j is a Lipschitz map which vanishes outside $N(v_j)$ and the functionals

$$\psi_j(u) = \frac{\rho_j(u)}{\sum_{i=1}^{n}\rho_i(u)} \tag{5.226}$$

define a partition of the unity subordinated to the covering $\{\rho_1,\dots,\rho_n\}$. Moreover, the mapping $p(u) = \sum_{j=1}^{n}\psi_j(u)v_j$ is continuous and maps K into itself, because of the convexity of K. Thus, by Brouwer's fixed point theorem, there exists u_0 in the convex closed hull of $\{v_1, \dots, v_n\}$ such that $p(u_0) = u_0$. Define

$$q(u) = \langle Au, p(u) - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \widehat{p(u)}(x) - \hat{u}(x))\,dx. \tag{5.227}$$

The convexity of the map $j^0(\hat{u}; \cdot)$ and the fact that $\sum_{j=1}^n \psi_j(u) = 1$ imply

$$q(u) \le \sum_{j=1}^n \psi_j(u)\langle Au, v_j - u\rangle + \sum_{j=1}^n \psi_j(u) \int_\Omega j^0(x, \hat{u}(x); \hat{v}_j(x) - \hat{u}(x))\,dx. \tag{5.228}$$

For arbitrary $u \in K$, there are only two possibilities: if $u \notin N(v_i)$, then $\psi_i(u) = 0$. On the other hand, for all $1 \le j \le n$ (there exists at least such an indice) such that $u \in N(v_j)$, we have $\psi_j(u) > 0$. Thus, by the definition of $N(v_j)$,

$$q(u) < 0, \quad \text{for every } u \in K. \tag{5.229}$$

But $q(u_0) = 0$, which gives a contradiction. □

The infinite dimensional version of Theorem 5.22 is stated in the following result.

Theorem 5.23. *Let K be a compact and convex subset of the infinite dimensional Banach space V and let j satisfy condition (j). If the operator $A : K \to V^*$ is w^*-demicontinuous, then problem (P) admits a solution.*

Remark 5.24. The condition of w^*-demicontinuity on the operator $A : K \to V^*$ in Theorem 5.23 may be replaced equivalently by the following assumption:

(A_1) the mapping $K \ni u \to \langle Au, v\rangle$ is weakly upper semicontinuous, for each $v \in V$.

Indeed, since on the compact set K the weak-topology is in fact the normed topology, we can replace equivalently the weak upper semicontinuity by upper semi-continuity. So we have to prove that the w^*-demicontinuity of A follows from assumption (A_1); but for any sequence $\{u_n\} \subset K$ converging to u one finds (by (A_1))

$$\begin{gathered} \limsup_{n\to\infty}\langle Au_n, v\rangle \le \langle Au, v\rangle, \\ \limsup_{n\to\infty}\langle Au_n, -v\rangle \le \langle Au, -v\rangle \Longleftrightarrow \liminf_{n\to\infty}\langle Au_n, v\rangle \ge \langle Au, v\rangle, \end{gathered} \tag{5.230}$$

for each fixed point $v \in V$. Thus, there exists $\lim_{n\to\infty}\langle Au_n, v\rangle$, and

$$\lim_{n\to\infty}\langle Au_n, v\rangle = \langle Au, v\rangle, \tag{5.231}$$

for every $v \in V$. Consequently, the sequence $\{Au_n\}$ converges to Au for the w^*-topology in V^*.

Remark 5.25. If A is w^*-demicontinuous, $\{u_n\} \subset K$ and $u_n \to u$, then

$$\lim_{n\to\infty} \langle Au_n, u_n \rangle = \langle Au, u \rangle. \tag{5.232}$$

This follows from the w^*-boundedness of $\{Au_n\}$ in V^* (as a w^*-convergent sequence) and from the fact that in real dual Banach spaces each w^*-bounded set is a (strongly) bounded set (this generally holds true in the topological dual of a real Hausdorff barrelled locally convex space, see Schaefer [206, Proposition IV.5.2]). Thus, in this case, one can write

$$\lim_{n\to\infty} \langle Au_n, v - u_n \rangle = \langle Au, v - u \rangle, \tag{5.233}$$

for each $v \in V$.

Proof of Theorem 5.23. Let F be an arbitrary finite dimensional subspace of V which intersects K. Let $i_{K\cap F}$ be the canonical injection of $K \cap F$ into K and i_F^* be the adjoint of the canonical injection i_F of F into V. We need the following auxiliary result. □

Lemma 5.26. *The operator*

$$B : K \cap F \longrightarrow F^*, \qquad B = i_F^* A i_{K\cap F} \tag{5.234}$$

is continuous.

Proof. We have to prove that the sequence $\{Bu_n\}$ converges to Bu in F^* for any sequence $\{u_n\} \subset K \cap F$ converging to u in $K \cap F$ (or in V). In order to do this, we prove that the sequence $\{Bu_n\}$ is weakly ($= w^*$) convergent to Bu, because F^* is a finite dimensional Banach space. Let V be an arbitrary point of F; then by the w^*-demicontinuity of the operator $A : K \to V^*$ it follows that

$$\begin{aligned}\langle Bu_n, v \rangle &= \langle i_F^* A i_{K\cap F} u_n, v \rangle = \langle i_F^* A u_n, v \rangle \\ &= \langle Au_n \cdot i_F, v \rangle = \langle Au_n, v \rangle \\ &\xrightarrow[n\to\infty]{} \langle Au, v \rangle = \langle Bu, v \rangle.\end{aligned} \tag{5.235}$$

Therefore, $\{Bu_n\}$ converges weakly to Bu. □

Remark 5.27. The above lemma also holds true if the operator A is continuous on finite dimensional subspaces of K.

Proof of Theorem 2.1 completed. For any $v \in K$, set

$$S(v) = \left\{ u \in K; \langle Au, v - u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0 \right\}. \tag{5.236}$$

Step 1. $S(v)$ is closed set.

We first observe that $S(v) \neq \emptyset$, since $v \in S(v)$. Let $\{u_n\} \subset S(v)$ be an arbitrary sequence which converges to u as $n \to \infty$. We have to prove that $u \in S(v)$. But, by (2), $u_n \in S(v)$ and by the part (a) of Lemma 5.18, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \left[\langle Au_n, v - u_n \rangle + \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x)) \right] dx \\ &= \lim_{n\to\infty} \langle Au_n, v - u_n \rangle + \limsup_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))\, dx \\ &\leq \langle Au, v - u \rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx. \end{aligned} \tag{5.237}$$

This is equivalent to $u \in S(v)$.

Step 2. The family $\{S(v); v \in K\}$ has the finite intersection property.

Let $\{v_1, \dots, v_n\}$ be an arbitrary finite subset of K and let F be the linear space spanned by this family. Applying Theorem 5.22 to the operator B defined in Lemma 5.26, we find $u \in K \cap F$ such that $u \in \cap_{j=1}^n S(v_j)$, which means that the family of closed sets $\{S(v); v \in K\}$ has the finite intersection property. But the set K is compact. Hence

$$\bigcap_{v \in K} S(v) \neq \emptyset, \tag{5.238}$$

which means that problem (P) has at least one solution. □

Weakening more the hypotheses on K by assuming that K is a closed, bounded and convex subset of the Banach space V, we need some more about the operators A and T with respect to the statement in Theorem 5.28. We first recall that an operator $A : K \to V^*$ is said to be monotone if, for every $u, v \in K$,

$$\langle Au - Av, u - v \rangle \geq 0. \tag{5.239}$$

Theorem 5.28. *Let V be a reflexive infinite dimensional Banach space and let $T : V \to L^p(\Omega, \mathbb{R}^k)$ be a linear and compact operator. Assume K is a closed, bounded, and convex subset of V and $A : K \to V^*$ is monotone and continuous on finite dimensional subspaces of K. If j satisfies condition (j) then problem (P) has at least one solution.*

Proof. Let F be an arbitrary finite dimensional subspace of V, which intersects K. Consider the canonical injections $i_{K\cap F} : K \cap F \to K$ and $i_F : F \to V$ and let $i_F^* : V^* \to F^*$ be the adjoint of i_F. Applying Theorem 5.22 to the continuous operator $B = i_F^\star A i_{K\cap F}$ (see Remark 5.27), we find some u_F in the compact set $K \cap F$ such that, for every $v \in K \cap F$,

$$\langle i_F^\star A i_{K\cap F} u_F, v - u_F \rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\, dx \geq 0. \tag{5.240}$$

But

$$0 \leq \langle Av - Au_F, v - u_F \rangle = \langle Av, v - u_F \rangle - \langle Au_F, v - u_F \rangle. \tag{5.241}$$

Hence, by relations (5.240), (5.241), and the observation that $\langle i_F^{\star} A i_{K\cap F} u_F, v - u_F\rangle = \langle Au_F, v - u_F\rangle$, we have

$$\langle Av, v - u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\,dx \geq 0, \tag{5.242}$$

for any $v \in K \cap F$. The set K is weakly closed as a closed convex set; thus it is weakly compact because it is bounded and V is a reflexive Banach-space.

Now, for every $v \in K$ define

$$M(v) = \left\{u \in K; \langle Av, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\,dx \geq 0\right\}. \tag{5.243}$$

The set $M(v)$ is weakly closed by the part (b) of Lemma 5.18 and by the fact that this set is weakly sequentially dense (see Holmes [106, pages 145–149]). We now show that the set $M = \cap_{v\in K} M(v) \subset K$ is nonempty. To prove this, it suffices to show that

$$\bigcap_{j=1}^{n} M(v_j) \neq \emptyset, \tag{5.244}$$

for any $v_1, \dots, v_n \in K$. Let F be the finite dimensional linear space spanned by $\{v_1, \dots, v_n\}$. Hence, by (5.242), there exists $u_F \in F$ such that, for every $v \in K \cap F$,

$$\langle Av, v - u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\,dx \geq 0. \tag{5.245}$$

This means that $u_F \in M(v_j)$, for every $1 \leq j \leq n$, which implies (5.244). Consequently, it follows that $M \neq \emptyset$. Therefore, there is some $u \in K$ such that, for every $v \in K$,

$$\langle Av, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\,dx \geq 0. \tag{5.246}$$

We argue in what follows that from (5.246) we can conclude that u is a solution of problem (P). Fix $w \in K$ and $\lambda \in (0,1)$. Putting $v = (1-\lambda)u + \lambda w \in K$ in (5.246) we find

$$\langle A((1-\lambda)u + \lambda w), \lambda(w - u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \lambda(\hat{w} - \hat{u})(x))\,dx \geq 0. \tag{5.247}$$

But $j^0(x, \hat{u}; \lambda\hat{v}) = \lambda j^0(x, \hat{u}; \hat{v})$, for any $\lambda > 0$. Therefore, (5.247) may be written, equivalently,

$$\langle A((1-\lambda)u + \lambda w), w - u\rangle + \int_\Omega j^0(x, \hat{u}(x); (\hat{w} - \hat{u})(x))\,dx \geq 0. \tag{5.248}$$

Let F be the vector space spanned by u and w. Taking into account the demi-continuity of the operator $A_{|K\cap F}$ and passing to the limit in (5.248) as $\lambda \to 0$, we obtain that u is a solution of problem (P). □

Remark 5.29. As the set $K \cap \{x \in V;\ \|u\| \leq R\}$ is a closed bounded and convex set in V, it follows from Theorem 5.28 that problem (5.249) in the formulation of our Theorem 5.30 has at least one solution for any fixed $R > 0$.

We conclude this section on Hartman-Stampacchia-type results with the following necessary and sufficient condition for the existence of a solution.

Theorem 5.30. *Assume that the same hypotheses as in Theorem 1.7 hold without the assumption of boundedness of K. Then a necessary and sufficient condition for the hemivariational inequality (P) to have a solution is that there exists $R > 0$ with the property that at least one solution of the problem*

$$u_R \in K \cap \{u \in V; \|u\| \le R\},$$
$$\langle Au_R, v - u_R\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \hat{v}(x) - \hat{u}_R(x))\,dx \ge 0, \tag{5.249}$$
$$\text{for every } v \in K \text{ with } \|v\| \le R,$$

satisfies the inequality $\|u_R\| < R$.

Proof. The necessity is evident.

Let us now suppose that there exists a solution u_R of (5.249) with $\|u_R\| < R$. We prove that u_R is a solution of (P). For any fixed $v \in K$, we choose $\varepsilon > 0$ small enough so that $w = u_R + \varepsilon(v - u_R)$ satisfies $\|w\| < R$. Hence, by (5.249)

$$\langle Au_R, \varepsilon(v - u_R)\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \varepsilon(\hat{v} - \hat{u}_R)(x))\,dx \ge 0 \tag{5.250}$$

and, using again the positive homogeneity of the map $v \mapsto j^0(u; v)$, the conclusion follows. □

5.5.2. Hartman-Stampacchia theory for variational-hemivariational inequalities

Let X denote a real reflexive Banach space, assume that (T, μ) is a measure space of positive and finite measure, and let $A : X \to X^\star$ be a nonlinear operator. We also assume that there are given $m \in \mathbb{N}$, $p \ge 1$ and a compact mapping $\gamma : X \to L^p(T, \mathbb{R}^m)$. As usually, we denote by p' the conjugated exponent of p.

Let $j : T \times \mathbb{R}^m \to \mathbb{R}$ be a function such that the mapping

$$j(\cdot, y) : T \longrightarrow \mathbb{R} \quad \text{is measurable, for every } y \in \mathbb{R}^m. \tag{5.251}$$

We assume that at least one of the following conditions hold: either there exists $k \in L^{p'}(T, \mathbb{R})$ such that

$$|\, j(x, y_1) - j(x, y_2)\,| \le k(x)\,|\,y_1 - y_2\,|, \quad \forall x \in T,\ \forall y_1, y_2 \in \mathbb{R}^m, \tag{5.252}$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz} \quad \forall x \in T, \tag{5.253}$$

and there exists $C > 0$ such that

$$|z| \le C(1 + |y|^{p-1}) \quad \forall x \in T,\ \forall y_1, y_2 \in \mathbb{R}^m,\ \forall z \in \partial_y j(x, y). \tag{5.254}$$

Let K be a nonempty closed, convex subset of X, $f \in X^\star$ and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that

$$D(\Phi) \cap K \neq \emptyset. \tag{5.255}$$

Consider the following inequality problem.

Find $u \in K$ such that for any $v \in K$,

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + \int_T j^0(x, \gamma(u(x)); \gamma(v(x) - u(x)))\,d\mu \geq 0, \tag{5.256}$$

where γ denotes the prescribed canonical mapping from X into $L^p(T, \mathbb{R}^m)$.

The following two situations are of particular interest in applications:

(i) $T = \Omega$, $\mu = dx$, $X = W^{1,q}(\Omega, \mathbb{R}^m)$ and $\gamma : X \to L^p(\Omega, \mathbb{R}^m)$, with $p < q^*$, is the Sobolev embedding operator;

(ii) $T = \partial\Omega$, $\mu = d\sigma$, $X = W^{1,p}(\Omega, \mathbb{R}^m)$ and $\gamma = i \circ \eta$, where $\eta : X \to W^{1-1/p,p}(\partial\Omega, \mathbb{R}^m)$ is the trace operator and $i : W^{1-1/p,p}(\partial\Omega, \mathbb{R}^m) \to L^p(\partial\Omega, \mathbb{R}^m)$ is the embedding operator.

We now recall the Knaster-Kuratowski-Mazurkiewicz (KKM, in short) principle (see [122] and [71, Theorem II.5.(1.2)]) which is equivalent with Brouwer's fixed point theorem and which plays a central role in the qualitative analysis of variational inequalities.

The KKM principle. Let S be a subset of the vector space X and assume that $G : S \to 2^X$ is a KKM map, that is,

$$\operatorname{Conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i), \tag{5.257}$$

for any finite subset $\{x_1, \ldots, x_n\}$ of S. We also assume that the set $G(x)$ is weakly compact for any $x \in S$.

Then the family $\{G(x);\ x \in S\}$ has the intersection property, that is,

$$\bigcap_{x \in S} G(x) \neq \emptyset. \tag{5.258}$$

The KKM principle enables us to prove the following basic auxiliary result.

Lemma 5.31. *Let K be a nonempty, bounded, closed, convex subset of X, $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex, lower semicontinuous functional such that* (5.255) *holds. Consider a Banach space Y such that there exists a linear and compact mapping $L : X \to Y$ and let $J : Y \to \mathbb{R}$ be an arbitrary locally Lipschitz function. Suppose in addition that the mapping $K \ni v \mapsto \langle Av, v - u\rangle$ is weakly lower semicontinuous, for every $u \in K$.*

Then, for every $f \in X^\star$, there exists $u \in K$ such that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(L(u), L(v - u)) \geq 0 \quad \forall v \in K. \tag{5.259}$$

Proof. Let us define the set-valued mapping $G : K \cap D(\Phi) \to 2^X$ by

$$G(x) = \{v \in K \cap D(\Phi); \langle Av - f, v - x\rangle - J^0(L(v); L(x) - L(v)) + \Phi(v) - \Phi(x) \leq 0\}. \tag{5.260}$$

We claim that the set $G(x)$ is weakly closed. Indeed, if $G(x) \ni v_n \rightharpoonup v$ then, by our hypotheses,

$$\begin{gathered}\langle Av, v - x\rangle \le \liminf_{n\to\infty} \langle Av_n, v_n - x\rangle,\\ \Phi(v) \le \liminf_{n\to\infty} \Phi(v_n).\end{gathered} \tag{5.261}$$

Moreover, $L(v_n) \to L(v)$ and thus, by the upper semicontinuity of J^0, we also obtain

$$\limsup_{n\to\infty} J^0(L(v_n); L(x - v_n)) \le J^0(L(v); L(x - v)). \tag{5.262}$$

Therefore,

$$-J^0(L(v); L(x - v)) \le \liminf_{n\to\infty} \big(- J^0(L(v_n); L(x - v_n))\big). \tag{5.263}$$

So, if $v_n \in G(x)$ and $v_n \rightharpoonup v$, then

$$\begin{aligned}&\langle Av - f, v - x\rangle - J^0(L(v); L(x - v)) + \Phi(v) - \Phi(x)\\ &\qquad \le \liminf \{\langle Av_n - f, v_n - x\rangle - J^0(L(v_n); L(v - v_n)) + \Phi(v_n) - \Phi(x)\} \le 0,\end{aligned} \tag{5.264}$$

which shows that $v \in G(x)$. Since K is bounded, it follows that $G(x)$ is weakly compact. This implies that

$$\bigcap_{x\in K\cap D(\Phi)} G(x) \ne \emptyset, \tag{5.265}$$

provided that the family $\{G(x); x \in K \cap D(\Phi)\}$ has the finite intersection property. We may conclude by using the KKM principle after showing that G is a KKM-mapping. Suppose by contradiction that there exist $x_1, \dots, x_n \in K \cap D(\Phi)$ and $y_0 \in \operatorname{Conv}\{x_1, \dots, x_n\}$ such that $y_0 \notin \bigcup_{i=1}^n G(x_i)$. Then

$$\langle Ay_0 - f, y_0 - x_i\rangle + \Phi(y_0) - \Phi(x_i) - J^0(L(y_0); L(x_i - y_0)) > 0 \quad \forall i = 1, \dots, n. \tag{5.266}$$

Therefore,

$$x_i \in \Lambda := \{x\in X; \langle Ay_0 - f, y_0 - x\rangle + \Phi(y_0) - \Phi(x) - J^0(L(y_0); L(x - y_0)) > 0\}, \tag{5.267}$$

for all $i \in \{1, \dots, n\}$. The set Λ is convex and thus $y_0 \in \Lambda$, leading to an obvious contradiction. So,

$$\bigcap_{x\in K\cap D(\Phi)} G(x) \ne \emptyset. \tag{5.268}$$

This yields an element $u \in K \cap D(\Phi)$ such that, for any $v \in K \cap D(\Phi)$,

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(L(u); L(v - u)) \ge 0. \tag{5.269}$$

This inequality is trivially satisfied if $v \notin D(\Phi)$ and the conclusion follows. □

We may now derive a result applicable to the inequality problem (5.256). Indeed, suppose that the above hypotheses are satisfied and set $Y = L^p(T, \mathbb{R}^m)$. Let $J : Y \to \mathbb{R}$ be the function defined by

$$J(u) = \int_T j(x, u(x))\, d\mu. \tag{5.270}$$

The conditions (5.252) or (5.253)-(5.254) on j ensure that J is locally Lipschitz on Y and

$$\int_T j^0(x, u(x); v(x))\, d\mu \geq J^0(u; v) \quad \forall u, v \in X. \tag{5.271}$$

It follows that

$$\int_T j^0(x, \gamma(u(x)); \gamma(v(x)))\, d\mu \geq J^0(\gamma(u); \gamma(v)) \quad \forall u, v \in X. \tag{5.272}$$

It results that if $u \in K$ is a solution of (5.259) then u solves the inequality problem (5.256), too. The following result follows.

Theorem 5.32. *Assume that the hypotheses of Lemma 5.31 are fulfilled for $Y = L^p(T, \mathbb{R}^m)$ and $L = \gamma$. Then problem* (5.256) *has at least a solution.*

In order to establish a variant of Lemma 5.31 for monotone and hemicontinuous operators, we need the following result which is due to Mosco (see [156]).

Mosco's theorem. Let K be a nonempty convex and compact subset of a topological vector space X. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $f, g : X \times X \to \mathbb{R}$ be two functions such that

(i) $g(x, y) \leq f(x, y)$, for every $x, y \in X$,
(ii) the mapping $f(\cdot, y)$ is concave, for any $y \in X$,
(iii) the mapping $g(x, \cdot)$ is lower semicontinuous, for every $x \in X$.

Let λ be an arbitrary real number. Then the following alternative holds:

(a) either there exists $y_0 \in D(\Phi) \cap K$ such that $g(x, y_0) + \Phi(y_0) - \Phi(x) \leq \lambda$, for any $x \in X$, or
(b) there exists $x_0 \in X$ such that $f(x_0, x_0) > \lambda$.

We notice that two particular cases of interest for the above result are if $\lambda = 0$ or $f(x, x) \leq 0$, for every $x \in X$.

Lemma 5.33. *Let K be a nonempty, bounded, closed subset of the real reflexive Banach space X and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex and lower semicontinuous function such that* (5.255) *holds. Consider a linear subspace Y of $X^\star$ such that there exists a linear and compact mapping $L : X \to Y$. Let $J : Y \to \mathbb{R}$ be a locally Lipschitz function. Suppose in addition that the operator $A : X \to X^\star$ is monotone and hemicontinuous.*

Then for each $f \in X^\star$, the inequality problem (5.259) *has at least a solution.*

Proof. Set

$$g(x,y) = \langle Ax - f, y - x\rangle - J^0(L(y); L(x) - L(y)),$$
$$f(x,y) = \langle Ay - f, y - x\rangle - J^0(L(y); L(x) - L(y)). \tag{5.273}$$

The monotonicity of A implies that

$$g(x,y) \le f(x,y) \quad \forall x, y \in X. \tag{5.274}$$

The mapping $x \mapsto f(x,y)$ is concave while the mapping $y \mapsto g(x,y)$ is weakly lower semicontinuous. Applying Mosco's theorem with $\lambda = 0$, we obtain the existence of $u \in K \cap D(\Phi)$ satisfying

$$g(w,u) + \Phi(u) - \Phi(w) \le 0 \quad \forall w \in K, \tag{5.275}$$

that is,

$$\langle Aw - f, w - u\rangle + \Phi(w) - \Phi(u) + J^0(L(u); L(w-u)) \ge 0 \quad \forall w \in K. \tag{5.276}$$

Fix $v \in K$ and set $w = u + \lambda(v - u) \in K$, for $\lambda \in [0,1)$. So, by (5.276),

$$\lambda\langle A(u+\lambda(v-u)) - f, v-u\rangle + \Phi(\lambda v + (1-\lambda)u) - \Phi(u) + J^0(L(u); \lambda L(v-u)) \ge 0. \tag{5.277}$$

Using the convexity of Φ, the fact that $J^0(u;\cdot)$ is positive homogeneous, and dividing then by $\lambda > 0$ we find

$$\langle A(\lambda v + (1-\lambda)u) - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(L(u); L(v-u)) \ge 0. \tag{5.278}$$

Now, taking $\lambda \to 0$ and using the hemicontinuity of A we find that u solves problem (5.259). ☐

The analogue of Theorem 5.32 for monotone and hemicontinuous operators is the following.

Theorem 5.34. *Assume that the hypotheses of Lemma 5.33 are fulfilled for $Y = L^p(T, \mathbb{R}^m)$ and $L = \gamma$. Then the inequality problem* (5.256) *admits at least a solution.*

5.5.3. Coercive variational-hemivariational inequalities

We observe that if j satisfies conditions (5.251) and (5.252) then, by the Cauchy-Schwarz inequality,

$$\left| \int_T j^0(x, \gamma(u(x)); \gamma(v(x)))\,d\mu \right| \le \int_T k(x)\,|\gamma(v(x))|\,d\mu$$
$$\le |k|_{p'} \cdot |\gamma(v)|_p \le C|k|_{p'}\|v\|, \tag{5.279}$$

where $|\cdot|_p$ denotes the norm in the space $L^p(T,\mathbb{R}^m)$ and $\|\cdot\|$ stands for the norm in X. On the other hand, if j satisfies conditions (5.251), (5.253), and (5.254), then

$$|j^0(x,\gamma(u(x));\gamma(v(x)))| \leq C\Big(1+|\gamma(u(x))|^{p-1}\Big)|\gamma(v(x))|. \tag{5.280}$$

Hence

$$\begin{aligned}\left|\int_T j^0(x,\gamma(u(x));\gamma(v(x)))d\mu\right| &\leq C\Big(|\gamma(v)|_1+|\gamma(u)|_{p-1}^{p}|\gamma(v)|_p\Big)\\ &\leq C_1\|v\|+C_2\|u\|^{p-1}\|v\|,\end{aligned} \tag{5.281}$$

for some suitable constants $C_1, C_2 > 0$. We discuss in this framework the solvability of coercive variational-hemivariational inequalities.

Theorem 5.35. *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbb{R}\cup\{+\infty\}$ a proper, convex and lower semicontinuous function such that $K\cap D(\Phi)\neq\emptyset$ an $A : X \to X^\star$ and operator such that the mapping $v \mapsto \langle Av, v-x\rangle$ is weakly lower semicontinuous, for all $x \in K$. The following holds:*

(i) *if j satisfies conditions* (5.251) *and* (5.252), *and if there exists $x_0 \in K\cap D(\Phi)$ such that*

$$\frac{\langle Aw, w-x_0\rangle+\Phi(w)}{\|w\|}\longrightarrow+\infty,\quad \text{as } \|w\|\longrightarrow+\infty, \tag{5.282}$$

then for each $f \in X^\star$, there exists $u \in K$ such that for any $v \in K$,

$$\langle Au-f, v-u\rangle+\Phi(v)-\Phi(u)+\int_T j^0(x,\gamma(u(x));\gamma(v(x))-\gamma(u(x)))d\mu\geq 0; \tag{5.283}$$

(ii) *if j satisfies conditions* (5.251), (5.253), *and* (5.254) *and if there exist $x_0 \in K\cap D(\Phi)$ and $\theta\geq p$ such that*

$$\frac{\langle Aw, w-x_0\rangle}{\|w\|^\theta}\longrightarrow+\infty,\quad \text{as } \|w\|\longrightarrow+\infty, \tag{5.284}$$

then for each $f \in X^\star$, there exists $u \in K$ satisfying (5.283).

Proof. There exists a positive integer n_0 such that

$$x_0\in K_n := \{x\in K; \|x\|\leq n\}\quad \forall n\geq n_0. \tag{5.285}$$

Applying Lemma 5.31 with J as defined in (5.270), we find some $u_n \in K_n$ such that, for every $n \geq n_0$ and any $v \in K_n$,

$$\langle Au_n-f, v-u_n\rangle+\Phi(v)-\Phi(u_n)+J^0(\gamma(u_n);\gamma(v)-\gamma(u_n))\geq 0. \tag{5.286}$$

We claim that the sequence (u_n) is bounded. Suppose by contradiction that $\|u_n\| \to +\infty$. Then, passing eventually to a subsequence, we may assume that

$$v_n := \frac{u_n}{||u_n||} \rightharpoonup v. \tag{5.287}$$

Setting $v = x_0$ in (5.286) and using (5.270), we obtain

$$\begin{aligned}\langle Au_n, u_n - x_0\rangle &+ \Phi(u_n)\\ &\leq \Phi(x_0) + \langle f, u_n - x_0\rangle + J^0(\gamma(u_n);\gamma(x_0) - u_n))\\ &\leq \Phi(x_0) + \langle f, u_n - x_0\rangle + \Big|\int_T j^0(x,\gamma(u_n);\gamma(x_0 - u_n))d\mu\Big|.\end{aligned} \tag{5.288}$$

Case (i). Using (5.279) we obtain

$$\langle Au_n, u_n - x_0\rangle + \Phi(u_n) \leq \Phi(x_0) + \langle f, u_n - x_0\rangle + c|k|_{p'}||u_n - x_0|| \tag{5.289}$$

and hence

$$\frac{\langle Au_n, u_n - x_0\rangle + \Phi(u_n)}{||u_n||} \leq \frac{\Phi(x_0)}{||u_n||} + \Big\langle f, v_n - x_0||u_n||^{-1}\Big\rangle + c|k|_{p'}\Big|\Big|v_n - x_0||u_n||^{-1}\Big|\Big|. \tag{5.290}$$

Passing to the limit as $n \to \infty$ we observe that the left-hand term in (5.290) tends to $+\infty$ while the right-hand term remains bounded which yields a contradiction.

Case (ii). The function Φ being convex and lower semicontinuous, we may apply the Hahn-Banach separation theorem to find that

$$\Phi(x) \geq \langle \alpha, x\rangle + \beta \quad \forall x \in X, \tag{5.291}$$

for some $\alpha \in X^\star$ and $\beta \in \mathbb{R}$. This means that

$$\Phi(x) \geq -\|\alpha\|_\star \|x\| + \beta \quad \forall x \in X. \tag{5.292}$$

From (5.288) and (5.281), we deduce that

$$\begin{aligned}\langle Au_n, u_n - x_0\rangle \leq \Phi(x_0) &+ \|\alpha\|_\star ||u_n|| - \beta + \langle f, u_n - x_0\rangle\\ &+ C_1||u_n - x_0|| + C_2||u_n||^{p-1}||u_n - x_0||.\end{aligned} \tag{5.293}$$

Thus

$$\begin{aligned}\frac{\langle Au_n, u_n - x_0\rangle}{||u_n||^\theta} \leq\ & \|\alpha\|_\star ||u_n||^{1-\theta} + (\Phi(x_0) - \beta)||u_n||^{-\theta}\\ &+ \Big\langle f, v_n||u_n||^{1-\theta} - x_0||u_n||^{-\theta}\Big\rangle\\ &+ C_1\Big|\Big|v_n||u_n||^{1-\theta} - x_0||u_n||^{-\theta}\Big|\Big|\\ &+ C_2||v_n - x_0||u_n||^{-1}\| \cdot ||u_n||^{p-\theta},\end{aligned} \tag{5.294}$$

and taking the limit as $n \to \infty$ we obtain a contradiction, since $\theta \geq p \geq 1$.

Thus, in both cases (i) and (ii), the sequence $\{u_n\}$ is bounded. This implies that, up to a subsequence, $u_n \rightharpoonup u \in K$. Let $v \in K$ be given. For all n large enough we have $v \in K_n$ and thus by (5.286),

$$\langle Au_n - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n);\gamma(v) - \gamma(u_n)) \leq 0. \tag{5.295}$$

Passing to the limit as $n \to \infty$ we obtain

$$\begin{gathered}
\langle Au - f, u - v\rangle \le \liminf_{n\to\infty} \langle Au_n - f, u_n - v\rangle, \\
\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n), \\
\gamma(u) = \lim_{n\to\infty} \gamma(u_n), \\
-J^0(\gamma(u); \gamma(v) - \gamma(u)) \le \liminf_{n\to\infty} \big(- J^0(\gamma(u_n); \gamma(v) - \gamma(u_n))\big).
\end{gathered} \tag{5.296}$$

Taking the inferior limit in (5.295), we obtain

$$\langle Au - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \le 0. \tag{5.297}$$

Since v has been chosen arbitrarily, we obtain

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \ge 0 \quad \forall v \in K. \tag{5.298}$$

Using now again (5.272), we conclude that u solves (5.283). □

The following result gives a corresponding variant for monotone hemicontinuous operators.

Theorem 5.36. *Let K be a nonempty closed convex subset of X, $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $A : X \to X^\star$ be a monotone and hemicontinuous operator. Assume* (5.282) *or* (5.284) *as in Theorem 5.35. Then the conclusions of Theorem 5.35 hold true.*

Proof. Using Lemma 5.33 we find a sequence $u_n \in K_n$ such that

$$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \ge 0 \quad \forall v \in K_n. \tag{5.299}$$

As in the proof of Theorem 5.35 we justify that $\{u_n\}$ is bounded and thus, up to a subsequence, we may assume that $u_n \rightharpoonup u$. By (5.299) and the monotonicity of A we deduce that

$$\langle Av - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) + J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \ge 0. \tag{5.300}$$

Let $v \in K$ be given. For n large enough, we obtain

$$\langle Av - f, u_n - v\rangle + \Phi(u_n) - \Phi(v) - J^0(\gamma(u_n); \gamma(v) - \gamma(u_n)) \le 0, \tag{5.301}$$

and taking the inferior limit we obtain

$$\langle Av - f, u - v\rangle + \Phi(u) - \Phi(v) - J^0(\gamma(u); \gamma(v) - \gamma(u)) \le 0. \tag{5.302}$$

Since v has been chosen arbitrarily it follows that

$$\langle Av - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \ge 0 \quad \forall v \in K. \tag{5.303}$$

Using now the same argument as in the proof of Lemma 5.33, we obtain that

$$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) + J^0(\gamma(u); \gamma(v) - \gamma(u)) \geq 0 \quad \forall v \in K \tag{5.304}$$

and the conclusion follows now by (5.272). □

5.6. Standing waves of multivalued Schrödinger equations

I don't like it, and I'm sorry I ever had anything to do with it.

Erwin Schrödinger talking about Quantum Physics

5.6.1. Physical motivation

In 1923, L. de Broglie recovers Bohr's formula for hydrogen atom by associating to each particle a wave of some frequency and identifying the stationary states of the electron to the stationary character of the wave. Independently and in the same year, Schrödinger proposes to express the Bohr's quantification conditions as an eigenvalue problem. The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear form of Schrödinger's equation is

$$\Delta\psi + \frac{8\pi^2 m}{\bar{h}^2}(E(x) - V(x))\psi = 0, \tag{5.305}$$

where ψ is the Schrödinger wave function, m is the mass, $\bar{h}$ denotes Planck's constant, E is the energy, and V stands for the potential energy. The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades now to a variety of areas in Mathematical Physics. The relevant fields of application may vary from optics and propagation of the electric field in optical fibers (Hasegawa and Kodama [102]), to the self-focusing and collapse of Langmuir waves in plasma physics (Zakharov [224]) and the behaviour of deep water waves and freak waves (the so-called rogue waves) in the ocean (Onorato, Osborne, Serio, and Bertone [172]). The nonlinear Schrödinger equation also describes various phenomena arising in: self-channelling of a high-power ultra-short laser in matter, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, in condensed matter theory, in plasma physics (e.g., the Kurihara superfluid film equation).

Schrödinger gives later the now classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He also develops a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proves the equivalence between his wave mechanics and Heisenberg's matrix, and introduces the time dependent Schrödinger's equation:

$$i\bar{h}\psi_t = -\frac{\bar{h}^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{5.306}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In physical problems, a cubic nonlinearity corresponding to $p = 3$ is common; in this case (5.306) is called the Gross-Pitaevskii equation. In the study of problem (5.306), Oh [171] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced in Kato [116]. Taking $\gamma > 0$ and $\bar{h} > 0$ sufficiently small and using a Lyapunov-Schmidt-type reduction, Oh [171] proved the existence of a standing wave solution of problem (5.306), that is, a solution of the form

$$\psi(x,t) = e^{-iEt/\bar{h}}u(x). \tag{5.307}$$

Note that substituting the ansatz (5.307) into (5.306) leads to

$$-\frac{\bar{h}^2}{2}\Delta u + \big(V(x) - E\big)u = |u|^{p-1}u. \tag{5.308}$$

The change of variable $y = \bar{h}^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2\big(V_{\bar{h}}(x) - E\big)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{5.309}$$

where $V_{\bar{h}}(x) = V(\bar{h}x)$.

If for some $\xi \in \mathbb{R}^N \setminus \{0\}$, $V(x + s\xi) = V(x)$ for all $s \in \mathbb{R}$, (5.306) is invariant under the Galilean transformation

$$\psi(x,t) \longmapsto \psi(x - \xi t, t)\exp\left(i\xi \cdot \frac{x}{\bar{h}} - \frac{1}{2}i|\xi|^2\frac{t}{\bar{h}}\right)\psi(x - \xi t, t). \tag{5.310}$$

Thus, in this case, standing waves reproduce solitary waves travelling in the direction of ξ. In other words, Schrödinger discovered that the standing waves are scalar waves rather than vector electromagnetic waves. This is an important difference, vector electromagnetic waves are mathematical waves which describe a direction (vector) of force, whereas the wave Motions of Space are scalar waves which are simply described by their wave-amplitude. The importance of this discovery was pointed out by Albert Einstein, who wrote “The Schrödinger method, which has in a certain sense the character of a field theory, does indeed deduce the existence of only discrete states, in surprising agreement with empirical facts. It does so on the basis of differential equations applying a kind of resonance argument...” (*On Quantum Physics*, 1954).

Motivated by the study of the propagation of pulse in nonlinear optical fiber, the nonlinear Schrödinger equation

$$-\Delta u + u = u^3 \quad \text{in } \mathbb{R}^3 \tag{5.311}$$

has been studied by many authors in the last few decades. It has been proved the existence of a ground state (least energy solution), which is radial with respect to some point, positive, and with exponential decay at infinity. In the same direction, Rabinowitz [187] proved that problem (5.309) has a ground state for $\bar{h} > 0$ small, under the assumption that $\inf_{x\in\mathbb{R}^N} V(x) > E$. After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{5.312}$$

under suitable conditions on a and assuming that f is smooth, superlinear and has a subcritical growth. A related equation has been considered in Lions [137], where it is studied the problem

$$\begin{gathered} -\Delta u + u = a(x)u^{p-1}, \\ u > 0 \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{5.313}$$

where $2 < p < 2N/(N-2)$ and $a(x) \geq a_\infty := \lim_{|x|\to\infty} a(x)$. The complementary case has been studied in Tintarev [219].

Our purpose in what follows is to study two multivalued versions of problem (5.312). We first consider a more general class of differential operators, the so-called $p(x)$-Laplace operators. This degenerate quasilinear operator is defined by $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ (where $p(x)$ is a certain function whose properties will be stated in what follows) and it generalizes the celebrated p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a constant. The $p(x)$-Laplace operator possesses more complicated nonlinearity than the p-Laplacian, for example, it is inhomogeneous. Next, we establish the existence of a weak solution for a class of nonlinear Schrödinger systems.

5.6.2. Schrödinger equations associated to nonhomogeneous differential operators

For any function $h(x,\cdot) \in L^\infty_{\text{loc}}(\mathbb{R})$ we denote by $\underline{h}$ (resp., $\overline{h}$) the lower (resp., upper) limit of h in its second variable, that is,

$$\begin{aligned} \underline{h}(x,t) &:= \lim_{\varepsilon\searrow 0} \operatorname{ess\,inf}\{h(x,s); |t-s| < \varepsilon\}, \\ \overline{h}(x,t) &= \lim_{\varepsilon\searrow 0} \operatorname{ess\,sup}\{h(x,s); |t-s| < \varepsilon\}. \end{aligned} \tag{5.314}$$

Let $a \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ be a variable potential such that, for some $a_0 > 0$,

$$a(x) \geq a_0 \quad \text{a.e. } x \in \mathbb{R}^N, \qquad \operatorname{ess}\lim_{|x|\to\infty} a(x) = +\infty. \tag{5.315}$$

Let $p : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 2)$ be a continuous function. Set $p^+ := \sup_{x\in\mathbb{R}^N} p(x)$ and $p^- := \inf_{x\in\mathbb{R}^N} p(x)$. We assume that p^+ is *finite*.

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for some $C > 0$, $q \in \mathbb{R}$ with $p^+ < q+1 \leq Np^-/(N-p^-)$ if $p^- < N$ and $p^+ < q+1 < +\infty$ if $p^- \geq N$, and $\mu > p^+$, we have

$$|f(x,t)| \leq C(|t| + |t|^q) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{5.316}$$

$$\lim_{\varepsilon\searrow 0} \operatorname{ess\,sup}\left\{\left|\frac{f(x,t)}{t^{p^+-1}}\right|; (x,t) \in \mathbb{R}^N \times (-\varepsilon,\varepsilon)\right\} = 0, \tag{5.317}$$

$$0 \leq \mu F(x,t) \leq s\underline{f}(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0,+\infty). \tag{5.318}$$

The hypothesis $q \leq Np^-/(N-p^-)$ enables us to allow an almost critical behaviour on f. One of the main features of our setting is that we do *not* assume that the nonlinearity f is continuous.

Let E denote the set of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $[a(x)]^{1/p(x)}u \in L^{p(x)}(\mathbb{R}^N)$ and $|\nabla u| \in L^{p(x)}(\mathbb{R}^N)$. Then E is a Banach space if it is endowed with the norm

$$\|u\|_E := \left| [a(x)]^{1/p(x)} u \right|_{p(x)} + |\nabla u|_{p(x)}. \tag{5.319}$$

We point out that E is continuously embedded in $W^{1,p(x)}(\mathbb{R}^N)$.

Bartsch, Liu and Weth [22] proved that if $p(x) \equiv 2$ and if the potential $a(x)$ fulfills more general hypotheses than (5.315), then the embedding $E \subset L^{q+1}(\mathbb{R}^N)$ is compact, whenever $2 \leq q < (N+2)/(N-2)$.

Open problem. Study if this embedding remains *compact* in our "variable exponent" framework and under assumption (5.315).

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between E^* and E.

Set $F(x,t) := \int_0^t f(x,s)ds$ and

$$\Psi(u) := \int_{\mathbb{R}^N} F(x, u(x))\,dx. \tag{5.320}$$

We observe that Ψ is locally Lipschitz on E. This follows by (5.316), Hölder's inequality and the continuous embedding $E \subset L^{q+1}(\mathbb{R}^N)$. Indeed, for all $u, v \in E$,

$$\left|\Psi(u) - \Psi(v)\right| \leq C\|u - v\|_E, \tag{5.321}$$

where $C = C(\|u\|_E, \|v\|_E) > 0$ depends only on $\max\{\|u\|_E, \|v\|_E\}$.

We are concerned in what follows with the multivalued elliptic problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + a(x)|u|^{p(x)-2}u \in [\underline{f}(x,u), \overline{f}(x,u)] \quad \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{5.322}$$

Definition 5.37. We say that $u \in E$ is a solution of problem (5.322) if $u \geq 0$, $u \not\equiv 0$, and $0 \in \partial I(u)$, where

$$I(u) := \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})dx - \int_{\mathbb{R}^N} F(x,u^+)dx \quad \forall u \in E. \tag{5.323}$$

The mapping $I : E \to \mathbb{R}$ is called the energy functional associated to problem (5.322). Our previous remarks show that I is locally Lipschitz on the Banach space E.

The above definition may be reformulated, equivalently, in terms of hemivariational inequalities. More precisely, $u \in E$ is a solution of (5.322) if $u \geq 0$, $u \not\equiv 0$ in $\mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} (|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv)\,dx + \int_{\mathbb{R}^N} (-F)^0(x,u;v)dx \geq 0, \tag{5.324}$$

for all $v \in E$.

Our main existence result is the following.

Theorem 5.38. *Assume that hypotheses* (5.315)–(5.318) *are fulfilled. Then problem* (5.322) *has at least one solution.*

Proof. We first claim that there exist positive constants C_1 and C_2 such that

$$f(x,t) \geq C_1 t^{\mu-1} - C_2 \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0,+\infty). \tag{5.325}$$

Indeed, by the definition of $\underline{f}$ we deduce that

$$\underline{f}(x,t) \leq f(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0,+\infty). \tag{5.326}$$

Set $\underline{F}(x,t) := \int_0^t \underline{f}(x,s)ds$. Thus, by our assumption (5.318),

$$0 \leq \mu \underline{F}(x,t) \leq t\underline{f}(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0,+\infty). \tag{5.327}$$

Next, by (5.327), there exist positive constants R and K_1 such that

$$\underline{F}(x,t) \geq K_1 t^{\mu} \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [R,+\infty). \tag{5.328}$$

Our claim (5.325) follows now directly by relations (5.326), (5.327), and (5.328).

Next, we observe that

$$\partial I(u) = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + a(x)|u|^{p(x)-2}u - \partial\Psi(u^+) \quad \text{in } E^*. \tag{5.329}$$

On the other hand, we have

$$\partial\Psi(u) \subset \left[\underline{f}(x,u(x)), \overline{f}(x,u(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N, \tag{5.330}$$

in the sense that if $w \in \partial\Psi(u)$ then

$$\underline{f}(x,u(x)) \leq w(x) \leq \overline{f}(x,u(x)) \quad \text{a.e. } x \in \mathbb{R}^N. \tag{5.331}$$

This means that if u_0 is a critical point of I, then there exists $w \in \partial\Psi(u_0)$ such that

$$-\operatorname{div}(|\nabla u_0|^{p(x)-2}\nabla u_0) + a(x)|u_0|^{p(x)-2}u_0 = w \quad \text{in } E^*. \tag{5.332}$$

This argument shows that, for proving Theorem 5.38, it is enough to show that the energy functional I has at least a nontrivial critical point $u_0 \in E$, $u_0 \geq 0$. We prove the existence of a solution of problem (5.322) by arguing that the hypotheses of Corollary 3.14 (mountain pass theorem for locally Lipschitz functionals) are fulfilled. More precisely, we check the following geometric assumptions:

$$I(0) = 0 \text{ and there exists } v \in E \text{ such that } I(v) \leq 0, \tag{5.333}$$

$$\text{there exist } \beta,\ \rho > 0 \text{ such that } I \geq \beta \quad \text{on } \{u \in E;\ \|u\|_E = \rho\}. \tag{5.334}$$

Verification of (5.333). Fix $w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ such that $w \geq 0$ in $\mathbb{R}^N$. In particular, we have

$$\int_{\mathbb{R}^N} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right)dx < +\infty. \tag{5.335}$$

So, by (5.325),

$$I(tw) = \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} (|\nabla w|^{p(x)} + a(x)w^{p(x)})\,dx - \Psi(tw)$$
$$\leq \frac{t^{p^+}}{p_-} \int_{\mathbb{R}^N} (|\nabla w|^{p(x)} + a(x)w^{p(x)})\,dx + C_2 t \int_{\mathbb{R}^N} w\,dx - C_1' t^{\mu} \int_{\mathbb{R}^N} w^{\mu}\,dx \quad \forall t > 0. \tag{5.336}$$

Since, by hypothesis, $1 < p^+ < \mu$, we deduce that $I(tw) < 0$ for $t > 0$ large enough.

Verification of (5.334). Our hypotheses (5.316) and (5.317) imply that, for any $\varepsilon > 0$, there exists some $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^q \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{5.337}$$

By (5.337) and Sobolev embeddings in variable exponent spaces we have, for any $u \in E$,

$$\Psi(u) \leq \varepsilon \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)}\,dx + \frac{A_\varepsilon}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}\,dx$$
$$\leq \varepsilon \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)}\,dx + C_4 \|u\|^{q+1}_{L^{q+1}(\mathbb{R}^N)}, \tag{5.338}$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus, by our hypotheses,

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})\,dx - \Psi(u^+)$$
$$\geq \frac{1}{p^+} \int_{\mathbb{R}^N} [|\nabla u|^{p(x)} + (a_0 - \varepsilon)|u|^{p(x)}]\,dx - C_4 \|u\|^{q+1}_{L^{q+1}(\mathbb{R}^N)} \geq \beta > 0, \tag{5.339}$$

for $\|u\|_E = \rho$, with ρ, ε, and β being small enough positive constants.

Denote

$$\mathcal{P} := \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) \neq 0,\ I(\gamma(1)) \leq 0\}$$
$$c := \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} I(\gamma(t)). \tag{5.340}$$

Set

$$\lambda_I(u) := \min_{\zeta \in \partial I(u)} \|\zeta\|_{E^*}. \tag{5.341}$$

So, by Corollary 3.14, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \longrightarrow c, \qquad \lambda_I(u_n) \longrightarrow 0. \tag{5.342}$$

Moreover, since $I(|u|) \leq I(u)$ for all $u \in E$, we can assume without loss of generality that $u_n \geq 0$ for every $n \geq 1$. So, for all positive integers n, there exists $\{w_n\} \in \partial\Psi(u_n) \subset E^*$ such that, for any $v \in E$,

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p(x)-2} \nabla u_n \nabla v + a(x) u_n^{p(x)-1} v)\,dx - \langle w_n, v\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.343}$$

Note that for all $u \in E$, $u \geq 0$, the definition of Ψ and our hypotheses yield

$$\Psi(u) \leq \frac{1}{\mu}\int_{\mathbb{R}^N} u(x)\underline{f}(x,u(x))\,dx. \tag{5.344}$$

Therefore, by (5.331), for every $u \in E$, $u \geq 0$, and for any $w \in \partial\Psi(u)$,

$$\Psi(u) \leq \frac{1}{\mu}\int_{\mathbb{R}^N} u(x)w(x)\,dx. \tag{5.345}$$

Hence

$$\begin{aligned}
I(u_n) &\geq \frac{\mu - p^+}{\mu p^+}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right)dx \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)} - w_n u_n\right)dx \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} w_n u_n\,dx - \Psi(u_n) \\
&\geq \frac{\mu - p^+}{\mu p^+}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right)dx \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)} - w_n u_n\right)dx \\
&= \frac{\mu - p^+}{\mu p^+}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right)dx \\
&\quad + \frac{1}{\mu}\langle -\Delta_{p(x)}u_n + au_n - w_n, u_n\rangle \\
&= \frac{\mu - p^+}{\mu p^+}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + a(x)u_n^{p(x)}\right)dx + o(1)\|u_n\|_E.
\end{aligned} \tag{5.346}$$

This relation and (5.342) show that the Palais-Smale sequence $\{u_n\}$ is bounded in E. It follows that $\{u_n\}$ converges weakly (up to a subsequence) in E and strongly in $L^{p(x)}_{\text{loc}}(\mathbb{R}^N)$ to some $u_0 \geq 0$. Taking into account that $w_n \in \partial\Psi(u_n)$ for all m, that $u_n \rightharpoonup u_0$ in E and that there exists $w_0 \in E^*$ such that $w_n \rightharpoonup w_0$ in E^* (up to a subsequence), we infer that $w_0 \in \partial\Psi(u_0)$. This follows from the fact that the map $u \mapsto F(x,u)$ is compact from E into L^1. Moreover, if we take $\phi \in C_c^\infty(\mathbb{R}^N)$ and let $\omega := \operatorname{supp}\phi$, then by (5.343) we get

$$\int_\omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0\nabla\varphi + a(x)u_0^{p(x)-1}\varphi - w_0\varphi\right)dx = 0. \tag{5.347}$$

So, by the definition of $(-F)^0$, we deduce that

$$\int_\omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0\nabla\varphi + a(x)u_0^{p(x)-1}\varphi\right)dx + \int_\omega (-F)^0(x,u_0;\varphi)\,dx \geq 0. \tag{5.348}$$

By density, this hemivariational inequality holds for all $\phi \in E$ and this means that u_0 solves problem (5.322).

It remains to prove that $u_0 \not\equiv 0$. If w_n is as in (5.343), then by (5.331) (recall that $u_n \geq 0$) and (5.342) (for large m) we deduce that

$$\begin{aligned} \frac{c}{2} &\leq I(u_n) - \frac{1}{p^-}\langle -\Delta_{p(x)}u_n + au_n - w_n, u_n\rangle \\ &= \frac{1}{p^-}\langle w_n, u_n\rangle - \int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\leq \frac{1}{p^-}\int_{\mathbb{R}^N} u_n\overline{f}(x,u_n)\,dx. \end{aligned} \tag{5.349}$$

Now, taking into account its definition, one deduces that $\overline{f}$ verifies (5.337), too. So, by (5.349), we obtain

$$0 < \frac{c}{2} \leq \frac{1}{p^-}\int_{\mathbb{R}^N}\left(\varepsilon u_n^2 + A_\varepsilon u_n^{q+1}\right)dx = \frac{\varepsilon}{p^-}||u_n||^2_{L^2(\mathbb{R}^N)} + \frac{A_\varepsilon}{p^-}||u_n||^{q+1}_{L^{q+1}(\mathbb{R}^N)}. \tag{5.350}$$

In particular, this shows that $\{u_n\}$ does not converge strongly to 0 in $L^{q+1}(\mathbb{R}^N)$. It remains to argue that $u_0 \not\equiv 0$. Since both $\|u_n\|_{L^{p^-}(\mathbb{R}^N)}$ and $\|\nabla u_n\|_{L^{p^-}(\mathbb{R}^N)}$ are bounded, it follows by Lemma I.1 in Lions [138] that the sequence $\{u_n\}$ "does not vanish" in $L^{p^-}(\mathbb{R}^N)$. Thus, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and $C > 0$ such that, for some $R > 0$,

$$\int_{z_n+B_R} u_n^{p^-}\,dx \geq C. \tag{5.351}$$

We claim that the sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$. Indeed, if not, up to a subsequence, it follows by (5.315) that

$$\int_{\mathbb{R}^N} a(x)u_n^{p^-}\,dx \longrightarrow +\infty \quad \text{as } n \longrightarrow \infty, \tag{5.352}$$

which contradicts our assumption $I(u_n) = c + o(1)$. Therefore, by (5.351), there exists an open bounded set $D \subset \mathbb{R}^N$ such that

$$\int_D u_n^{p^-}\,dx \geq C > 0. \tag{5.353}$$

In particular, this relation implies that $u_0 \not\equiv 0$ and our proof is concluded. □

5.6.3. Coupled Schrödinger elliptic systems with discontinuous nonlinearity

Coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers or Kerr-like photorefractive media in optics. For instance, Menyuk [145] showed that the evolution of two orthogonal pulse envelopes in birefringent optical fibers is governed by the following coupled nonlinear Schrödinger system:

$$\begin{aligned} i\phi_t + \phi_{xx} + \phi^3 + \beta\psi^3 &= 0, \\ i\psi_t + \psi_{xx} + \psi^3 + \beta\phi|^3 &= 0, \end{aligned} \tag{5.354}$$

where β is a positive constant depending on the anisotropy of the fibers. If we are looking for *standing wave solutions* of (5.354), namely solutions of the form

$$\phi(x,t) = e^{iw_1^2 t}u(x), \qquad \psi(x,t) = e^{iw_2^2 t}v(x), \tag{5.355}$$

then problem (5.354) reduces to

$$\begin{aligned} -u_{xx} + u &= u^3 + \beta v^2 u \quad \text{in } \mathbb{R} \\ -v_{xx} + w_2^2 v/w_1^2 &= v^3 + \beta u^2 v \quad \text{in } \mathbb{R}. \end{aligned} \tag{5.356}$$

Problem (5.356) has been studied in higher dimensions by Cipolatti and Zumpichiatti [44]. By concentration compactness arguments they prove the existence and the regularity of a nontrivial ground state solution. Recently, Lin and Wei [136] have proved the existence of least energy positive solutions in arbitrary dimensions and they have argued that if $\beta < 0$ then problem (5.356) does not have a ground state solution.

Another motivation for the study of coupled Schrödinger systems arises from the Hartree-Fock theory for the double condensate, that is a binary mixture of Bose-Einstein condensates in two different hyperfine states, cf. Esry et. all. [78]. Coupled nonlinear Schrödinger systems are also important for industrial applications in fiber communications systems (see Hasegawa and Kodama [102]) and all-optical switching devices (see Islam [112]).

Motivated by the above results, we consider the following class of coupled Schrödinger stationary systems in $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{aligned} -\Delta u_1 + a(x)u_1 &= f(x,u_1,u_2) \quad \text{in } \mathbb{R}^N, \\ -\Delta u_2 + b(x)u_2 &= g(x,u_1,u_2) \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{5.357}$$

We assume that $a, b \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and there exist $\underline{a}, \underline{b} > 0$ such that

$$a(x) \geq \underline{a}, \qquad b(x) \geq \underline{b} \quad \text{a.e. in } \mathbb{R}^N, \tag{5.358}$$

and $\operatorname{ess\,lim}_{|x|\to\infty} a(x) = \operatorname{ess\,lim}_{|x|\to\infty} b(x) = +\infty$. Our aim in what follows is to study the existence of solutions to the above problem in the case when f, g are not continuous functions. Our goal is to show how variational methods can be used to find existence results for stationary nonsmooth Schrödinger systems.

We suppose that $f(x,\cdot,\cdot), g(x,\cdot,\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Denote:

$$\begin{aligned} \underline{f}(x,t_1,t_2) &= \lim_{\delta\to 0} \operatorname{ess\,inf}\{f(x,s_1,s_2);\, |t_i - s_i| \leq \delta;\ i = 1,2\}, \\ \overline{f}(x,t_1,t_2) &= \lim_{\delta\to 0} \operatorname{ess\,sup}\{f(x,s_1,s_2);\, |t_i - s_i| \leq \delta;\ i = 1,2\}, \\ \underline{g}(x,t_1,t_2) &= \lim_{\delta\to 0} \operatorname{ess\,inf}\{g(x,s_1,s_2);\, |t_i - s_i| \leq \delta;\ i = 1,2\}, \\ \overline{g}(x,t_1,t_2) &= \lim_{\delta\to 0} \operatorname{ess\,sup}\{g(x,s_1,s_2);\, |t_i - s_i| \leq \delta;\ i = 1,2\}. \end{aligned} \tag{5.359}$$

Under these conditions, we reformulate problem (5.357) as follows:

$$-\Delta u_1 + a(x)u_1 \in \left[\underline{f}(x,u_1(x),u_2(x)),\overline{f}(x,u_1(x),u_2(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N,$$
$$-\Delta u_2 + b(x)u_2 \in \left[\underline{g}(x,u_1(x),u_2(x)),\overline{g}(x,u_1(x),u_2(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N. \tag{5.360}$$

Let $H^1 = H(\mathbb{R}^N,\mathbb{R}^2)$ denote the Sobolev space of all $U = (u_1,u_2) \in (L^2(\mathbb{R}^N))^2$ with weak derivatives $\partial u_1/\partial x_j$, $\partial u_2/\partial x_j$ $(j = 1,\dots,N)$ also in $L^2(\mathbb{R}^N)$, endowed with the usual norm

$$\begin{aligned}\|U\|_{H_1}^2 &= \int_{\mathbb{R}^N}(|\nabla U|^2 + |U|^2)\,dx\\ &= \int_{\mathbb{R}^N}(|\nabla u_1|^2 + |\nabla u_2|^2 + u_1^2 + u_2^2)\,dx.\end{aligned} \tag{5.361}$$

Given the functions $a,b : \mathbb{R}^N \to \mathbb{R}$ as above, define the function space

$$E = \left\{U = (u_1,u_2) \in H^1; \int_{\mathbb{R}^N}\left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right)dx < +\infty\right\}. \tag{5.362}$$

Then the space E endowed with the norm

$$\|U\|_E^2 = \int_{\mathbb{R}^N}\left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right)dx \tag{5.363}$$

becomes a Hilbert space.

Since $a(x) \geq \underline{a} > 0, b(x) \geq \underline{b} > 0$, we have the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N,\mathbb{R}^2)$ for all $2 \leq q \leq 2^* = 2N/(N-2)$.

We assume throughout the section that $f,g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are nontrivial measurable functions satisfying the following hypotheses:

$$\begin{aligned}|f(x,t)| &\leq C(|t| + |t|^p) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2,\\ |g(x,t)| &\leq C(|t| + |t|^p) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R}^2,\end{aligned} \tag{5.364}$$

where $p < 2^*$;

$$\begin{aligned}&\lim_{\delta\to 0} \operatorname{ess\,sup}\left\{\frac{|f(x,t)|}{|t|}; (x,t) \in \mathbb{R}^N \times (-\delta,+\delta)^2\right\} = 0\\ &\lim_{\delta\to 0} \operatorname{ess\,sup}\left\{\frac{|g(x,t)|}{|t|}; (x,t) \in \mathbb{R}^N \times (-\delta,+\delta)^2\right\} = 0;\end{aligned} \tag{5.365}$$

f and g are chosen so that the mapping $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,t_1,t_2) := \int_0^{t_1} f(x,\tau,t_2)d\tau + \int_0^{t_2} g(x,0,\tau)d\tau$ satisfies

$$\begin{gathered}F(x,t_1,t_2) = \int_0^{t_2} g(x,t_1,\tau)d\tau + \int_0^{t_1} f(x,\tau,0)d\tau,\\ F(x,t_1,t_2) = 0 \quad \text{if and only if } t_1 = t_2 = 0;\end{gathered} \tag{5.366}$$

there exists $\mu > 2$ such that for any $x \in \mathbb{R}^N$,

$$0 \leq \mu F(x,t_1,t_2) \leq \begin{cases} t_1\underline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2), & t_1,\ t_2 \geq 0, \\ t_1\underline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2), & t_1 \geq 0,\ t_2 \leq 0, \\ t_1\overline{f}(x,t_1,t_2) + t_2\overline{g}(x,t_1,t_2), & t_1,\ t_2 \leq 0, \\ t_1\overline{f}(x,t_1,t_2) + t_2\underline{g}(x,t_1,t_2), & t_1 \leq 0,\ t_2 \geq 0. \end{cases} \tag{5.367}$$

Definition 5.39. A function $U = (u_1,u_2) \in E$ is called solution of problem (5.360) if there exists a function $W = (w_1,w_2) \in L^2(\mathbb{R}^N,\mathbb{R}^2)$ such that

(i)

$$\begin{aligned} \underline{f}(x,u_1(x),u_2(x)) &\leq w_1(x) \leq \overline{f}(x,u_1(x),u_2(x)) \quad \text{a.e. } x \in \mathbb{R}^N, \\ \underline{g}(x,u_1(x),u_2(x)) &\leq w_2(x) \leq \overline{g}(x,u_1(x),u_2(x)) \quad \text{a.e. } x \in \mathbb{R}^N, \end{aligned} \tag{5.368}$$

(ii)

$$\int_{\mathbb{R}^N} (\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1v_1 + b(x)u_2v_2)\,dx = \int_{\mathbb{R}^N} (w_1v_1 + w_2v_2)\,dx \quad \forall (v_1,v_2) \in E. \tag{5.369}$$

Our main existence result is the following.

Theorem 5.40. *Assume that conditions* (5.364)–(5.367) *are fulfilled. Then problem* (5.360) *has at least a nontrivial solution in E.*

Before giving the proof we deduce some auxiliary results.

Let Ω be an arbitrary domain in $\mathbb{R}^N$. Set

$$E_\Omega = \left\{U = (u_1,u_2) \in H^1(\Omega;\mathbb{R}^2); \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + au_1^2 + bu_2^2\right) < +\infty\right\} \tag{5.370}$$

which is endowed with the norm

$$\|U\|_{E_\Omega}^2 = \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + a(x)u_1^2 + b(x)u_2^2\right)dx. \tag{5.371}$$

Then E_Ω becomes a Hilbert space.

Lemma 5.41. *The functional $\Psi_\Omega : E_\Omega \to \mathbb{R}$ defined by $\Psi_\Omega(U) = \int_\Omega F(x,U)\,dx$ is locally Lipschitz on E_Ω.*

Proof. We first observe that

$$\begin{aligned} F(x,U) &= F(x,u_1,u_2) \\ &= \int_0^{u_1} f(x,\tau,u_2)\,d\tau + \int_0^{u_2} g(x,0,\tau)\,d\tau \\ &= \int_0^{u_2} g(x,u_1,\tau)\,d\tau + \int_0^{u_1} f(x,\tau,0)\,d\tau \end{aligned} \tag{5.372}$$

is a Carathéodory functional which is locally Lipschitz with respect to the second variable. Indeed, by (5.364),

$$\begin{aligned} |F(x,t_1,t) - F(x,s_1,t)| &= \left| \int_{s_1}^{t_1} f(x,\tau,t)\,d\tau \right| \\ &\leq \left| \int_{s_1}^{t_1} C\big(|\tau,t| + |\tau,t|^p\big)\,d\tau \right| \\ &\leq k(t_1,s_1,t)\,|t_1 - s_1|. \end{aligned} \tag{5.373}$$

Similarly,

$$|F(x,t,t_2) - F(x,t,s_2)| \leq k(t_2,s_2,t)\,|t_2 - s_2|. \tag{5.374}$$

Therefore,

$$\begin{aligned} |F(x,t_1,t_2)-F(x,s_1,s_2)| &\leq |F(x,t_1,t_2) - F(x,s_1,t_2)| + |F(x,t_1,s_2)-F(x,s_1,s_2)| \\ &\leq k(V)\,|(t_2,s_2) - (t_1,s_1)|, \end{aligned} \tag{5.375}$$

where V is a neighborhood of (t_1,t_2), (s_1,s_2).

Set

$$\chi_1(x) = \max\{u_1(x),v_1(x)\}, \quad \chi_2(x) = \max\{u_2(x),v_2(x)\} \quad \forall x \in \Omega. \tag{5.376}$$

It is obvious that if $U = (u_1,u_2)$, $V = (v_1,v_2)$ belong to E_Ω, then $(\chi_1,\chi_2) \in E_\Omega$. So, by Hölder's inequality and the continuous embedding $E_\Omega \subset L^p(\Omega;\mathbb{R}^2)$,

$$|\Psi_\Omega(U) - \Psi_\Omega(V)| \leq C\Big(\|\chi_1,\chi_2\|_{E_\Omega}\Big)\|U - V\|_{E_\Omega}, \tag{5.377}$$

which concludes the proof. □

The following result is a generalization of Lemma 6 in Mironescu and Rădulescu [154].

Lemma 5.42. *Let Ω be an arbitrary domain in $\mathbb{R}^N$ and let $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a Borel function such that $f(x,.) \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. Then $\underline{f}$ and $\overline{f}$ are Borel functions.*

Proof. Since the requirement is local, we may suppose that f is bounded by M and it is nonnegative. Denote

$$f_{m,n}(x,t_1,t_2) = \left(\int_{t_1-1/n}^{t_1+1/n} \int_{t_2-1/n}^{t_2+1/n} |f(x,s_1,s_2)|^m ds_1\, ds_2 \right)^{1/m}. \tag{5.378}$$

Since $\overline{f}(x,t_1,t_2) = \lim_{\delta\to 0} \operatorname{ess\,sup}\{f(x,s_1,s_2);\ |t_i - s_i| \le \delta;\ i = 1,2\}$ we deduce that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}^*$ such that for $|t_i - s_i| < 1/n$ $(i = 1,2)$ we have $|\operatorname{ess\,sup} f(x,s_1,s_2) - \overline{f}(x,t_1,t_2)| < \varepsilon$ or, equivalently,

$$\overline{f}(x,t_1,t_2) - \varepsilon < \operatorname{ess\,sup} f(x,s_1,s_2) < \overline{f}(x,t_1,t_2) + \varepsilon. \tag{5.379}$$

By the second inequality in (5.379) we obtain

$$f(x,s_1,s_2) \le \overline{f}(x,t_1,t_2) + \varepsilon \quad \text{a.e. } x \in \Omega \text{ for } |t_i - s_i| < \frac{1}{n}\ (i = 1,2) \tag{5.380}$$

which yields

$$f_{m,n}(x,t_1,t_2) \le (\overline{f}(x,t_1,t_2) + \varepsilon)\left(\sqrt{4/n^2}\right)^{1/m}. \tag{5.381}$$

Let

$$A = \left\{ (s_1,s_2) \in \mathbb{R}^2;\ |t_i - s_i| < \frac{1}{n}\ (i = 1,2);\ \overline{f}(x,t_1,t_2) - \varepsilon \le f(x,s_1,s_2) \right\}. \tag{5.382}$$

By the first inequality in (5.379) and the definition of the essential supremum we obtain that $|A| > 0$ and

$$f_{m,n} \le \left(\iint_A (f(x,s_1,s_2))^m ds_1\, ds_2 \right)^{1/m} \ge (\overline{f}(x,s_1,s_2) - \varepsilon)|A|^{1/m}. \tag{5.383}$$

Since (5.381) and (5.383) imply

$$\overline{f}(x,t_1,t_2) = \lim_{n\to\infty} \lim_{m\to\infty} f_{m,n}(x,t_1,t_2), \tag{5.384}$$

it suffices to prove that $f_{m,n}$ is a Borel function. Let

$$\begin{gathered} \mathcal{M} = \{f : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R};\ |f| \le M,\ f \text{ is a Borel function}\}, \\ \mathcal{N} = \{f \in \mathcal{M};\ f_{m,n} \text{ is a Borel function}\}. \end{gathered} \tag{5.385}$$

Cf. Berberian [24, page 178]; $\mathcal{M}$ is the smallest set of functions having the following properties:

(i) $\{f \in C(\Omega \times \mathbb{R}^2;\mathbb{R});\ |f| \le M\} \subset \mathcal{M}$;

(ii) $f^{(k)} \in \mathcal{M}$ and $f^{(k)} \xrightarrow{k} f$ imply $f \in \mathcal{M}$.

Since $\mathcal{N}$ contains obviously the continuous functions and (ii) is also true for $\mathcal{N}$ then, by the Lebesgue dominated convergence theorem, we obtain that $\mathcal{M} = N$. For $\underline{f}$ we note that $\underline{f} = -(-\overline{f})$ and the proof of Lemma 5.42 is complete. □

Let us now assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain. By the continuous embedding $L^{p+1}(\Omega;\mathbb{R}^2) \hookrightarrow L^2(\Omega;\mathbb{R}^2)$, we may define the locally Lipschitz functional $\Psi_\Omega : L^{p+1}(\Omega;\mathbb{R}^2) \to \mathbb{R}$ by $\Psi_\Omega(U) = \int_\Omega F(x,U)dx$.

Lemma 5.43. *Under the above assumptions and for any $U \in L^{p+1}(\Omega;\mathbb{R}^2)$, one has*

$$\partial\Psi_\Omega(U)(x) \subset \left[\underline{f}(x,U(x)),\overline{f}(x,U(x))\right] \times \left[\underline{g}(x,U(x)),\overline{g}(x,U(x))\right] \quad \text{a.e. } x \in \Omega, \tag{5.386}$$

in the sense that if $W = (w_1,w_2) \in \partial\Psi_\Omega(U) \subset L^{p+1}(\Omega;\mathbb{R}^2)$ then

$$\underline{f}(x,U(x)) \le w_1(x) \le \overline{f}(x,U(x)) \quad \text{a.e. } x \in \Omega, \tag{5.387}$$

$$\underline{g}(x,U(x)) \le w_2(x) \le \overline{g}(x,U(x)) \quad \text{a.e. } x \in \Omega. \tag{5.388}$$

Proof. By the definition of the Clarke gradient we have

$$\int_\Omega (w_1v_1 + w_2v_2)dx \le \Psi_\Omega^0(U,V) \quad \forall V = (v_1,v_2) \in L^{p+1}(\Omega;\mathbb{R}^2). \tag{5.389}$$

Choose $V = (v,0)$ such that $v \in L^{p+1}(\Omega)$, $v \ge 0$ a.e. in Ω. Thus, by Lemma 5.42,

$$\begin{aligned}\int_\Omega w_1 v &\le \limsup_{\substack{(h_1,h_2)\to U\\ \lambda\searrow 0}} \frac{\int_\Omega\left(\int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))d\tau\right)dx}{\lambda}\\ &\le \int_\Omega\left(\limsup_{\substack{(h_1,h_2)\to U\\ \lambda\searrow 0}} \frac{1}{\lambda}\int_{h_1(x)}^{h_1(x)+\lambda v(x)} f(x,\tau,h_2(x))d\tau\right)dx\\ &\le \int_\Omega \overline{f}(x,u_1(x),u_2(x))v(x)dx.\end{aligned} \tag{5.390}$$

Analogously, we obtain

$$\int_\Omega \underline{f}(x,u_1(x),u_2(x))v(x)dx \le \int_\Omega w_1 v\,dx \quad \forall v \ge 0 \text{ in } \Omega. \tag{5.391}$$

Arguing by contradiction, suppose that (5.387) is false. Then there exist $\varepsilon > 0$, a set $A \subset \Omega$ with $|A| > 0$ and w_1 as above such that

$$w_1(x) > \overline{f}(x,U(x)) + \varepsilon \quad \text{in } A. \tag{5.392}$$

Taking $v = \mathbf{1}_A$ in (5.390) we obtain

$$\int_\Omega w_1 v\,dx = \int_A w_1 dx \le \int_A \overline{f}(x,U(x))dx, \tag{5.393}$$

which contradicts (5.392). Proceeding in the same way, we obtain the corresponding result for g in (5.388). □

By Lemma 5.43 and the embedding $E_\Omega \hookrightarrow L^{p+1}(\Omega, \mathbb{R}^2)$ we obtain also that for $\Psi_\Omega : E_\Omega \to \mathbb{R}$, $\Psi_\Omega(U) = \int_\Omega F(x, U)dx$ we have

$$\partial\Psi_\Omega(U)(x) \subset \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right] \quad \text{a.e. } x \in \Omega. \tag{5.394}$$

Let $V \in E_\Omega$. Then $\widetilde{V} \in E$, where $\widetilde{V} : \mathbb{R}^N \to \mathbb{R}^2$ is defined by

$$\widetilde{V} = \begin{cases} \widetilde{V}(x), & x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{5.395}$$

For $W \in E^*$ we consider $W_\Omega \in E_\Omega^*$ such that $\langle W_\Omega, V\rangle = \langle W, \widetilde{V}\rangle$ for all V in E_Ω. Set $\Psi : E \to \mathbb{R}$, $\Psi(U) = \int_{\mathbb{R}^N} F(x, U)$.

Lemma 5.44. *Let $W \in \partial\Psi(U)$, where $U \in E$. Then $W_\Omega \in \partial\Psi_\Omega(U)$, in the sense that $W_\Omega \in \partial\Psi_\Omega(U|_\Omega)$.*

Proof. By the definition of the Clarke gradient we deduce that $\langle W, \widetilde{V}\rangle \le \Psi^0(U, \widetilde{V})$ for all V in E_Ω,

$$\begin{aligned}
\Psi^0(U, \widetilde{V}) &= \limsup_{\substack{H\to U, H\in E\\ \lambda\to 0}} \frac{\Psi(H + \lambda\widetilde{V}) - \Psi(H)}{\lambda} \\
&= \limsup_{\substack{H\to U, H\in E\\ \lambda\to 0}} \frac{\int_{\mathbb{R}^N} \left(F(x, H + \lambda\widetilde{V}) - F(x, H)\right) dx}{\lambda} \\
&= \limsup_{\substack{H\to U, H\in E\\ \lambda\to 0}} \frac{\int_{\Omega} \left(F(x, H + \lambda\widetilde{V}) - F(x, H)\right) dx}{\lambda} \\
&= \limsup_{\substack{H\to U, H\in E_\Omega\\ \lambda\to 0}} \frac{\int_{\Omega} \left(F(x, H + \lambda\widetilde{V}) - F(x, H)\right) dx}{\lambda} \\
&= \Psi_\Omega^0(U, V).
\end{aligned} \tag{5.396}$$

Hence $\langle W_\Omega, V\rangle \le \Psi_\Omega^0(U, V)$ which implies $W_\Omega \in \partial\Psi_\Omega^0(U)$. □

By Lemmas 5.43 and 5.44 we obtain that for any $W \in \partial\Psi(U)$ (with $U \in E$), W_Ω satisfies (5.387) and (5.388). We also observe that for $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ we have $W_{\Omega_1}|_{\Omega_1\cap\Omega_2} = W_{\Omega_2}|_{\Omega_1\cap\Omega_2}$.

Let $W_0 : \mathbb{R}^N \to \mathbb{R}$, where $W_0(x) = W_\Omega(x)$ if $x \in \Omega$. Then W_0 is well defined and

$$W_0(x) \in \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N, \tag{5.397}$$

and, for all $\phi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$, $\langle W, \phi\rangle = \int_{\mathbb{R}^N} W_0\phi$. By density of $C_c^\infty(\mathbb{R}^N, \mathbb{R}^2)$ in E we deduce that $\langle W, V\rangle = \int_{\mathbb{R}^N} W_0 V\, dx$ for all V in E. Hence, for a.e. $x \in \mathbb{R}^N$,

$$W(x) = W_0(x) \in \left[\underline{f}(x, U(x)), \overline{f}(x, U(x))\right] \times \left[\underline{g}(x, U(x)), \overline{g}(x, U(x))\right]. \tag{5.398}$$

Proof of Theorem 5.35. Define the energy functional $I : E \to \mathbb{R}$ by

$$\begin{aligned} I(U) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\nabla u_1\right|^2 + \left|\nabla u_2\right|^2 + a(x)u_1^2 + b(x)u_2^2\right)dx - \int_{\mathbb{R}^N} F(x,U)dx \\ &= \frac{1}{2}\|U\|_E^2 - \Psi(U). \end{aligned} \tag{5.399}$$

□

The existence of solutions to problem (5.360) will be justified by Corollary 3.14 (the nonsmooth variant of the mountain pass theorem) applied to the functional I, even if the Palais-Smale condition is not fulfilled. More precisely, we check the following geometric hypotheses:

$$I(0) = 0 \text{ and there exists } V \in E \text{ such that } I(V) \le 0; \tag{5.400}$$

$$\text{there exist } \beta, \rho > 0 \text{ such that } I \ge \beta \text{ on } \{U \in E;\ \|U\|_E = \rho\}. \tag{5.401}$$

Verification of (5.400). It is obvious that $I(0) = 0$. For the second assertion we need the following lemma.

Lemma 5.45. *There exist two positive constants C_1 and C_2 such that*

$$f(x,s,0) \ge C_1 s^{\mu-1} - C_2 \quad \text{for a.e. } x \in \mathbb{R}^N;\ s \in [0,+\infty). \tag{5.402}$$

Proof. We first observe that (5.367) implies

$$0 \le \mu F(x,s,0) \le \begin{cases} s\underline{f}(x,s,0) & \text{if } s \in [0,+\infty), \\ s\overline{f}(x,s,0) & \text{if } s \in (-\infty,0]. \end{cases} \tag{5.403}$$

Verification of (5.400) *continued.* Choose $v \in C_c^\infty(\mathbb{R}^N) - \{0\}$ so that $v \ge 0$ in $\mathbb{R}^N$. We have $\int_{\mathbb{R}^N}(|\nabla v|^2 + a(x)v^2)dx < \infty$, hence $t(v,0) \in E$ for all $t \in \mathbb{R}$. Thus, by Lemma 5.45 we obtain

$$\begin{aligned} I(t(v,0)) &= \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + a(x)v^2)dx - \int_{\mathbb{R}^N}\int_0^{tv} f(x,\tau,0)d\tau \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + a(x)v^2)dx - \int_{\mathbb{R}^N}\int_0^{tv}(C_1\tau^{\mu-1} - C_2)d\tau \\ &= \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + a(x)v^2)dx + C_2 t\int_{\mathbb{R}^N} v\,dx - C_1' t^\mu \int_{\mathbb{R}^N} v^\mu dx < 0 \end{aligned} \tag{5.404}$$

for $t > 0$ large enough.

Verification of (5.401). We observe that (5.365), (5.366), and (5.367) imply that, for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that

$$\left.\begin{aligned} \left|f(x,s)\right| &\le \varepsilon|s| + A_\varepsilon|s|^p \\ \left|g(x,s)\right| &\le \varepsilon|s| + A_\varepsilon|s|^p \end{aligned}\right. \quad \text{for a.e. } (x,s) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{5.405}$$

By (5.405) and Sobolev's embedding theorem we have, for any $U \in E$,

$$\begin{aligned} |\Psi(U)| &= |\Psi(u_1,u_2)| \\ &\leq \int_{\mathbb{R}^N}\int_0^{|u_1|} |f(x,\tau,u_2)|\,d\tau + \int_{\mathbb{R}^N}\int_0^{u_2} |g(x,0,\tau)|\,d\tau \\ &\leq \int_{\mathbb{R}^N}\left(\frac{\varepsilon}{2}|(u_1,u_2)|^2 + \frac{A_\varepsilon}{p+1}|(u_1,u_2)|^{p+1}\right)dx \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{\varepsilon}{2}|u_2|^2 + \frac{A_\varepsilon}{p+1}|u_2|^{p+1}\right)dx \\ &\leq \varepsilon\|U\|_{L^2}^2 + \frac{2A_\varepsilon}{p+1}\|U\|_{L^{p+1}}^{p+1} \\ &\leq \varepsilon C_3\|U\|_E^2 + C_4\|U\|_E^{p+1}, \end{aligned} \tag{5.406}$$

where ε is arbitrary and $C_4 = C_4(\varepsilon)$. Thus

$$I(U) = \frac{1}{2}\|U\|_E^2 - \Psi(U) \geq \frac{1}{2}\|U\|_E^2 - \varepsilon C_3\|U\|_E^2 - C_4\|U\|_E^{p+1} \geq \beta > 0, \tag{5.407}$$

for $\|U\|_E = \rho$, with ρ, ε, and β sufficiently small positive constants.

Denote

$$\begin{gathered} \mathcal{P} = \{\gamma \in C([0,1],E);\ \gamma(0)=0,\ \gamma(1)\neq 0,\ I(\gamma(1))\leq 0\}, \\ c = \inf_{\gamma\in\mathcal{P}}\max_{t\in[0,1]} I(\gamma(t)). \end{gathered} \tag{5.408}$$

Set

$$\lambda_I(U) = \min_{\xi\in\partial I(U)}\|\xi\|_{E^*}. \tag{5.409}$$

Thus, by Corollary 3.14, there exists a sequence $\{U_M\} \subset E$ such that

$$I(U_m) \longrightarrow c, \qquad \lambda_I(U_m) \longrightarrow 0. \tag{5.410}$$

So, there exists a sequence $\{W_m\} \subset \partial\Psi(U_m)$, $W_m = (w_m^1, w_m^2)$, such that

$$\left(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + a(x)u_m^2 - w_m^2\right) \longrightarrow 0 \quad \text{in } E^*. \tag{5.411}$$

Note that, by (5.367),

$$\begin{aligned} &\Psi(U) \\ &\quad\leq \frac{1}{\mu}\left(\int_{u_1\geq 0} u_1\underline{f}(x,U) + \int_{u_1\leq 0} u_1\overline{f}(x,U) + \int_{u_2\geq 0} u_1\underline{g}(x,U) + \int_{u_2\leq 0} u_2\overline{g}(x,U)\right). \end{aligned} \tag{5.412}$$

Therefore, by (5.398),

$$\Psi(U) \leq \frac{1}{\mu}\int_{\mathbb{R}^N} U(x)W(x)dx = \frac{1}{\mu}\int_{\mathbb{R}^N}(u_1w_1 + u_2w_2)dx, \tag{5.413}$$

for every $U \in E$ and $W \in \partial\Psi(U)$. Hence, if $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E, we have

$$
\begin{aligned}
I(U_m) &= \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N}\left(|\nabla u_m^1|^2+|\nabla u_m|^2+a(x)|u_m|^1+b(x)|u_m|^2\right)dx \\
&\quad + \frac{1}{\mu}\langle(-\Delta u_m^1+a(x)u_m^1-w_m^1, -\Delta u_m^2+b(x)u_m^2-w_m^2), U_m\rangle \\
\frac{1}{\mu}\langle W_m, U_m\rangle - \Psi(U_m) &\geq \frac{\mu-2}{2\mu}\int_{\mathbb{R}^N}\left(|\nabla u_m^1|^2+|\nabla u_m^2|^2+a(x)|u_m^1|^2+b(x)|u_m^2|^2\right)dx \\
&\quad + \frac{1}{\mu}\langle(-\Delta u_m^1+a(x)u_m^1-w_m^1, -\Delta u_m^2+b(x)u_m^2-w_m^2), U_m\rangle \\
&\geq \frac{\mu-2}{2\mu}\|U_m\|_E^2 - o(1)\|U_m\|_E.
\end{aligned}
\tag{5.414}
$$

This relation in conjunction with (5.410) implies that the Palais-Smale sequence $\{U_m\}$ is bounded in E. Thus, it converges weakly (up to a subsequence) in E and strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$ to some U. Taking into account that $W_m \in \partial\Psi(U_m)$ and $U_m \rightharpoonup U$ in E, we deduce from (5.411) that there exists $W \in E^*$ such that $W_m \rightharpoonup W$ in E^* (up to a subsequence). Since the mapping $U \mapsto F(x,U)$ ia compact from E to L^1, it follows that $W \in \partial\Psi(U)$. Therefore,

$$
\begin{aligned}
W(x) &\in \left[\underline{f}(x,U(x)), \overline{f}(x,U(x))\right] \times \left[\underline{g}(x,U(x)), \overline{g}(x,U(x))\right] \quad \text{a.e. } x \in \mathbb{R}^N, \\
&(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2) = 0 \\
&\quad \Longleftrightarrow \int_{\mathbb{R}^N}(\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + a(x)u_1v_1 + b(x)u_2v_2)\,dx \\
&\quad = \int_{\mathbb{R}^N}(w_1v_1 + w_2v_2)\,dx,
\end{aligned}
\tag{5.415}
$$

for all $(v_1, v_2) \in E$. These last two relations show that U is a solution of problem (5.360).

It remains to prove that $U \not\equiv 0$. If $\{W_m\}$ is as in (5.411), then by (5.367), (5.398), (5.410), and for large m,

$$
\begin{aligned}
\frac{c}{2} &\leq I(U_m) - \frac{1}{2}\langle(-\Delta u_m^1 + a(x)u_m^1 - w_m^1, -\Delta u_m^2 + b(x)u_m^2 - w_m^2), U_m\rangle \\
&= \frac{1}{2}\langle W_m, U_m\rangle - \int_{\mathbb{R}^N} F(x,U_m)\,dx \\
&\leq \frac{1}{2}\left(\int_{u_1\geq 0} u_1\underline{f}(x,U) + \int_{u_1\leq 0} u_1\overline{f}(x,U) + \int_{u_2\geq 0} u_1\underline{g}(x,U) + \int_{u_2\leq 0} u_2\overline{g}(x,U)\right).
\end{aligned}
\tag{5.416}
$$

Now, taking into account the definition of $\overline{f}$, $\underline{f}$, $\overline{g}$, $\underline{g}$, we deduce that these functions verify (5.400), too. So, by (5.416),

$$\frac{c}{2} \le \int_{\mathbb{R}^N} \left(\varepsilon |U_m|^2 + A_\varepsilon |u_m|^{p+1}\right) dx = \varepsilon \|U_m\|_{L^2}^2 + A_\varepsilon \|U_m\|_{L^{p+1}}^{p+1}. \tag{5.417}$$

Thus, $\{U_m\}$ does not converge strongly to 0 in $L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$. Next, since $\{U_m\}$ is bounded in $E \subset L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$, it follows that $\{U_m\}$ and $\{\nabla U_m\}$ are bounded in $L^{p+1}(\mathbb{R}^N;\mathbb{R}^2)$. So, by [138, Lemma I.1], there exist a sequence $\{z_m\}$ and positive numbers C, R such that, for all $m \ge 1$,

$$\int_{z_m + B_R} \left[(u_m^1)^2 + (u_m^2)\right] dx \ge C. \tag{5.418}$$

The next step consists in showing that $\{z_m\}$ is bounded. Arguing by contradiction and using (5.358) we obtain, up to a subsequence,

$$\int_{\mathbb{R}^N} \left[a(x)(u_m^1)^2 + b(x)(u_m^2)^2\right] dx \longrightarrow +\infty \quad \text{as } m \longrightarrow \infty. \tag{5.419}$$

But this relation contradicts our assumption $I(U_m) \to c$. So, by (5.418), there exists an open bounded set $D \subset \mathbb{R}^N$ such that

$$\int_D \left[a(x)(u_m^1)^2 + b(x)(u_m^2)^2\right] dx \ge C. \tag{5.420}$$

This relation implies that $U \not\equiv 0$, which concludes the proof. □

5.7. Bibliographical notes

The existence result formulated in Theorem 5.5 is due to Mironescu and Rădulescu [153]. The forced pendulum problem (5.78)–(5.81) was studied by Mawhin and Willem [143] and the multivalued version established in Theorem 5.9 is proved by Mironescu and Rădulescu [153]. Existence results for multivalued Landesman-Lazer problems have been established by Rădulescu [193, 194]. Inequality problems of hemivariational type have been studied starting with the pioneering papers by Panagiotopoulos [175, 164]. The Hartman–Stampacchia theory for problems of this type (Theorems 5.22–5.30 in this section) was developed by Fundos et al. [83]. The existence theorems 5.32–5.36 are due to Motreanu and Rădulescu [159, 160]. Standing waves for the Schrödinger equation have been studied in a nonsmooth setting by Gazzola and Rădulescu [84] and Dinu [68].

Bibliography

[1] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications*, vol. 141 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2001.

[2] R. P. Agarwal and D. O'Regan, *Singular Differential and Integral Equations with Applications*, Kluwer Academic, Dordrecht, The Netherlands, 2003.

[3] R. P. Agarwal, D. O'Regan, and R. Precup, "Construction of upper and lower solutions with applications to singular boundary value problems," *Journal of Computational Analysis and Applications*, vol. 7, no. 2, pp. 205–221, 2005.

[4] S. Ahmad, A. C. Lazer, and J. L. Paul, "Elementary critical point theory and perturbations of elliptic boundary value problems at resonance," *Indiana University Mathematics Journal*, vol. 25, no. 10, pp. 933–944, 1976.

[5] M. Aizenman and B. Simon, "Brownian motion and Harnack inequality for Schrödinger operators," *Communications on Pure and Applied Mathematics*, vol. 35, no. 2, pp. 209–273, 1982.

[6] S. Alama and G. Tarantello, "On the solvability of a semilinear elliptic equation via an associated eigenvalue problem," *Mathematische Zeitschrift*, vol. 221, no. 3, pp. 467–493, 1996.

[7] H. Amann, "On the existence of positive solutions of nonlinear elliptic boundary value problems," *Indiana University Mathematics Journal*, vol. 21, pp. 125–146, 1971.

[8] H. Amann, "Multiple positive fixed points of asymptotically linear maps," *Journal of Functional Analysis*, vol. 17, pp. 174–213, 1974.

[9] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, no. 4, pp. 620–709, 1976.

[10] H. Amann and M. G. Crandall, "On some existence theorems for semi-linear elliptic equations," *Indiana University Mathematics Journal*, vol. 27, no. 5, pp. 779–790, 1978.

[11] A. Ambrosetti, "Critical points and nonlinear variational problems," *Mémoires de la Société Mathématique de France*, no. 49, p. 139, 1992.

[12] A. Ambrosetti and G. Mancini, "Theorems of existence and multiplicity for nonlinear elliptic problems with noninvertible linear part," *Annali della Scuola Normale Superiore di Pisa*, vol. 5, no. 1, pp. 15–28, 1978.

[13] A. Ambrosetti and G. Mancini, "Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance the case of the simple eigenvalue," *Journal of Differential Equations*, vol. 28, no. 2, pp. 220–245, 1978.

[14] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of Functional Analysis*, vol. 14, no. 4, pp. 349–381, 1973.

[15] A. Ambrosetti and M. Struwe, "Existence of steady vortex rings in an ideal fluid," *Archive for Rational Mechanics and Analysis*, vol. 108, no. 2, pp. 97–109, 1989.

[16] D. Arcoya, "Periodic solutions of Hamiltonian systems with strong resonance at infinity," *Differential and Integral Equations*, vol. 3, no. 5, pp. 909–921, 1990.

[17] V. I. Arnold, *Lectures on Partial Differential Equations*, Universitext, Springer, Berlin, Germany, 2004.

[18] J. P. Aubin and F. H. Clarke, "Shadow prices and duality for a class of optimal control problems," *SIAM Journal on Control and Optimization*, vol. 17, no. 5, pp. 567–586, 1979.

[19] C. Bandle, "Asymptotic behaviour of large solutions of quasilinear elliptic problems," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 54, no. 5, pp. 731–738, 2003.

[20] V. Barbu, *Partial Differential Equations and Boundary Value Problems*, vol. 441 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.

[21] P. Bartolo, V. Benci, and D. Fortunato, "Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 9, pp. 981–1012, 1983.

[22] T. Bartsch, Z. Liu, and T. Weth, "Sign changing solutions of superlinear Schrödinger equations," *Communications in Partial Differential Equations*, vol. 29, no. 1-2, pp. 25–42, 2004.

[23] V. Benci and P. H. Rabinowitz, "Critical point theorems for indefinite functionals," *Inventiones Mathematicae*, vol. 52, no. 3, pp. 241–273, 1979.

[24] S. K. Berberian, *Measure and Integration*, Macmillan, New York, NY, USA, 1967.

[25] M. S. Berger and E. Podolak, "On the solutions of a nonlinear Dirichlet problem," *Indiana University Mathematics Journal*, vol. 24, pp. 837–846, 1975.

[26] G. D. Birkhoff, "Dynamical systems with two degrees of freedom," *Transactions of the American Mathematical Society*, vol. 18, no. 2, pp. 199–300, 1917.

[27] M. Bocher, "The smallest characteristic numbers in a certain exceptional case," *Bulletin of the American Mathematical Society*, vol. 21, no. 1, pp. 6–9, 1914.

[28] K. Borsuk, "Drei Sätze über die *n*-dimensionale Euklidische Sphäre," *Fundamenta Mathematicae*, vol. 21, pp. 177–190, 1933.

[29] H. Brézis, *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland Mathematics Studies, no. 5. Notas de Matemática (50), North-Holland, Amsterdam, The Netherlands, 1973.

[30] H. Brézis, *Analyse Fonctionnelle. Théorie, Méthodes et Applications*, Masson, Paris, France, 1992.

[31] H. Brézis, J.-M. Coron, and L. Nirenberg, "Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz," *Communications on Pure and Applied Mathematics*, vol. 33, no. 5, pp. 667–684, 1980.

[32] H. Brézis and S. Kamin, "Sublinear elliptic equations in $\mathbb{R}^N$," *Manuscripta Mathematica*, vol. 74, no. 1, pp. 87–106, 1992.

[33] H. Brézis and L. Nirenberg, "Remarks on finding critical points," *Communications on Pure and Applied Mathematics*, vol. 44, no. 8-9, pp. 939–963, 1991.

[34] H. Brézis and L. Nirenberg, "Nonlinear Functional Analysis and Applications to Partial Differential Equations," in preparation.

[35] H. Brézis and L. Oswald, "Remarks on sublinear elliptic equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 10, no. 1, pp. 55–64, 1986.

[36] R. F. Brown, *A Topological Introduction to Nonlinear Analysis*, Birkhäuser, Boston, Mass, USA, 2nd edition, 2004.

[37] N. P. Cac, A. M. Fink, and J. A. Gatica, "Nonnegative solutions of the radial Laplacian with nonlinearity that changes sign," *Proceedings of the American Mathematical Society*, vol. 123, no. 5, pp. 1393–1398, 1995.

[38] L. Čaklović, S. Li, and M. Willem, "A note on Palais-Smale condition and coercivity," *Differential and Integral Equations*, vol. 3, no. 4, pp. 799–800, 1990.

[39] J. Chabrowski, *Variational Methods for Potential Operator Equations with Applications to Nonlinear Elliptic Equations*, vol. 24 of *de Gruyter Studies in Mathematics*, Walter de Gruyter, Berlin, Germany, 1997.

[40] J. Chabrowski, *Weak Convergence Methods for Semilinear Elliptic Equations*, World Scientific, Singapore, 1999.

[41] K. C. Chang, "Variational methods for non-differentiable functionals and their applications to partial differential equations," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 1, pp. 102–129, 1981.

[42] M. Chipot, *Elements of Nonlinear Analysis*, Birkhäuser Advanced Texts, Birkhäuser, Basel, Switzerland, 2000.

[43] G. Čobanov, A. Marino, and D. Scolozzi, "Multiplicity of eigenvalues for the Laplace operator with respect to an obstacle, and nontangency conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 15, no. 3, pp. 199–215, 1990.

[44] R. Cipolatti and W. Zumpichiatti, "On the existence and regularity of ground states for a nonlinear system of coupled Schrödinger equations in $\mathbb{R}^N$," *Computational and Applied Mathematics*, vol. 18, no. 1, pp. 15–29, 1999.

[45] F.-C. Ş. Cîrstea and V. D. Rădulescu, "Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbb{R}^N$," *Journal of Mathematical Analysis and Applications*, vol. 229, no. 2, pp. 417–425, 1999.

[46] F. H. Clarke, "Generalized gradients and applications," *Transactions of the American Mathematical Society*, vol. 205, pp. 247–262, 1975.

[47] F. H. Clarke, "Generalized gradients of Lipschitz functionals," *Advances in Mathematics*, vol. 40, no. 1, pp. 52–67, 1981.

[48] P. Clément and G. Sweers, "Getting a solution between sub- and supersolutions without monotone iteration," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, vol. 19, no. 2, pp. 189–194, 1987.

[49] C. V. Coffman, "A minimum-maximum principle for a class of non-linear integral equations," *Journal d'Analyse Mathématique*, vol. 22, pp. 391–419, 1969.

[50] D. S. Cohen and T. W. Laetsch, "Nonlinear boundary value problems suggested by chemical reactor theory," *Journal of Differential Equations*, vol. 7, no. 2, pp. 217–226, 1970.

[51] J.-N. Corvellec, M. Degiovanni, and M. Marzocchi, "Deformation properties for continuous functionals and critical point theory," *Topological Methods in Nonlinear Analysis*, vol. 1, no. 1, pp. 151–171, 1993.

[52] D. G. Costa and J. V. A. Gonçalves, "Critical point theory for nondifferentiable functionals and applications," *Journal of Mathematical Analysis and Applications*, vol. 153, no. 2, pp. 470–485, 1990.

[53] D. G. Costa and E. A. De B. Silva, "The Palais-Smale condition versus coercivity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 4, pp. 371–381, 1991.

[54] M. G. Crandall and P. H. Rabinowitz, "Bifurcation from simple eigenvalues," *Journal of Functional Analysis*, vol. 8, no. 2, pp. 321–340, 1971.

[55] M. G. Crandall and P. H. Rabinowitz, "Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems," *Archive for Rational Mechanics and Analysis*, vol. 58, no. 3, pp. 207–218, 1975.

[56] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, "On a Dirichlet problem with a singular nonlinearity," *Communications in Partial Differential Equations*, vol. 2, no. 2, pp. 193–222, 1977.

[57] P. G. de Gennes, "Wetting: statics and dynamics," *Reviews of Modern Physics*, vol. 57, no. 3, pp. 827–863, 1985.

[58] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, Boston, Mass, USA, 2nd edition, 2005.

[59] E. De Giorgi, A. Marino, and M. Tosques, "Problemi di evoluzione in spazi metrici e curve di massima pendenza," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 68, no. 3, pp. 180–187, 1980.

[60] M. Degiovanni and S. Lancelotti, "Perturbations of even nonsmooth functionals," *Differential and Integral Equations*, vol. 8, no. 5, pp. 981–992, 1995.

[61] M. Degiovanni, M. Marzocchi, and V. D. Rădulescu, "Multiple solutions of hemivariational inequalities with area-type term," *Calculus of Variations and Partial Differential Equations*, vol. 10, no. 4, pp. 355–387, 2000.

[62] M. Degiovanni and V. D. Rădulescu, "Perturbations of nonsmooth symmetric nonlinear eigenvalue problems," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 329, no. 4, pp. 281–286, 1999.

[63] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Elliptic Equations*, vol. 106 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1985.

[64] J. I. Díaz and J. E. Saá, "Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 305, no. 12, pp. 521–524, 1987.

[65] E. DiBenedetto, *Partial Differential Equations*, Birkhäuser, Boston, Mass, USA, 1995.

[66] T.-L. Dinu, "Entire solutions of the nonlinear eigenvalue logistic problem with sign-changing potential and absorption," *Communications on Pure and Applied Analysis*, vol. 2, no. 3, pp. 311–321, 2003.

[67] T.-L. Dinu, "Entire solutions of sublinear elliptic equations in anisotropic media," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 382–392, 2006.

[68] T.-L. Dinu, "Standing wave solutions of Schrödinger systems with discontinuous nonlinearity in anisotropic media," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 73619, 13 pages, 2006.

[69] Y. Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations. Maximum Principles and Applications*, vol. 2 of *Series in Partial Differential Equations and Applications*, World Scientific, Hackensack, NJ, USA, 2006.

[70] J. Dugundji, "An extension of Tietze's theorem," *Pacific Journal of Mathematics*, vol. 1, pp. 353–367, 1951.

[71] J. Dugundji and A. Granas, *Fixed Point Theory, Vol. I.*, vol. 61 of *Monografie Matematyczne*, Polish Scientific, Warsaw, Poland, 1982.

[72] L. Dupaigne, M. Ghergu, and V. D. Rădulescu, "Lane-Emden-Fowler equations with convection and singular potential," *Journal des Mathématiques Pures et Appliqués*, vol. 87, no. 6, pp. 563–581, 2007.

[73] A. L. Edelson, "Entire solutions of singular elliptic equations," *Journal of Mathematical Analysis and Applications*, vol. 139, no. 2, pp. 523–532, 1989.

[74] Yu. V. Egorov and M. A. Shubin, *Foundations of the Classical Theory of Partial Differential Equations*, Springer, Berlin, Germany, 1998.

[75] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 47, no. 2, pp. 324–353, 1974.

[76] I. Ekeland, "Nonconvex minimization problems," *Bulletin of the American Mathematical Society*, vol. 1, no. 3, pp. 443–474, 1979.

[77] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, vol. 1 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 1976.

[78] B. D. Esry, C. H. Greene, J. P. Burke Jr., and J. L. Bohn, "Hartree-Fock theory for double condensates," *Physical Review Letters*, vol. 78, no. 19, pp. 3594–3597, 1997.

[79] L. C. Evans, *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 1998.

[80] R. A. Fisher, "The wave of advance of advantageous genes," *Annals of Eugenics*, vol. 7, pp. 355–369, 1937.

[81] L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, vol. 128 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2000.

[82] S. Fucik, J. Necas, J. Soucek, and V. Soucek, *Spectral Analysis of Nonlinear Operators*, vol. 346 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1973.

[83] P. D. Panagiotopoulos, M. Fundo, and V. D. Rădulescu, "Existence theorems of Hartman-Stampacchia type for hemivariational inequalities and applications," *Journal of Global Optimization*, vol. 15, no. 1, pp. 41–54, 1999.

[84] F. Gazzola and V. D. Rădulescu, "A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^n$," *Differential and Integral Equations*, vol. 13, no. 1–3, pp. 47–60, 2000.

[85] I. M. Gel'fand, "Some problems in the theory of quasi-linear equations," *Uspekhi Matematicheskikh Nauk*, vol. 14, no. 2, pp. 87–158, 1959.

[86] M. Ghergu and V. D. Rădulescu, "Bifurcation and asymptotics for the Lane-Emden-Fowler equation," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 337, no. 4, pp. 259–264, 2003.

[87] M. Ghergu and V. D. Rădulescu, "Sublinear singular elliptic problems with two parameters," *Journal of Differential Equations*, vol. 195, no. 2, pp. 520–536, 2003.

[88] M. Ghergu and V. D. Rădulescu, "Explosive solutions of semilinear elliptic systems with gradient term," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A*, vol. 97, no. 3, pp. 467–475, 2003.

[89] M. Ghergu and V. D. Rădulescu, "Existence and nonexistence of entire solutions to the logistic differential equation," *Abstract and Applied Analysis*, vol. 2003, no. 17, pp. 995–1003, 2003.

[90] M. Ghergu and V. D. Rădulescu, "Nonradial blow-up solutions of sublinear elliptic equations with gradient term," *Communications on Pure and Applied Analysis*, vol. 3, no. 3, pp. 465–474, 2004.

[91] M. Ghergu and V. D. Rădulescu, "Bifurcation for a class of singular elliptic problems with quadratic convection term," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 338, no. 11, pp. 831–836, 2004.

[92] M. Ghergu and V. D. Rădulescu, "Multi-parameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 135, no. 1, pp. 61–83, 2005.

[93] M. Ghergu and V. D. Rădulescu, "On a class of sublinear singular elliptic problems with convection term," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 635–646, 2005.

[94] M. Ghergu and V. D. Rădulescu, "Singular elliptic problems with lack of compactness," *Annali di Matematica Pura ed Applicata*, vol. 185, no. 1, pp. 63–79, 2006.

[95] M. Ghergu and V. D. Rădulescu, "Singular elliptic problems with convection term in anisotropic media," in *Mathematical Analysis and Applications: International Conference on Mathematical Analysis and Applications. AIP Conference Proceedings*, C. Niculescu and V. D. Rădulescu, Eds., vol. 835, pp. 74–89, American Institute of Physics, Melville, NY, USA, 2006.

[96] M. Ghergu and V. D. Rădulescu, "On a class of singular Gierer-Meinhardt systems arising in morphogenesis," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 344, no. 3, pp. 163–168, 2007.

[97] M. Ghergu and V. D. Rădulescu, "On the influence of a subquadratic convection term in singular elliptic problems," in *Applied Analysis and Differential Equations*, O. Cârjă and I. Vrabie, Eds., pp. 127–138, World Scientific, Hackensack, NJ, USA, 2007.

[98] N. Ghoussoub and D. Preiss, "A general mountain pass principle for locating and classifying critical points," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 6, no. 5, pp. 321–330, 1989.

[99] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, vol. 224 of *Fundamental Principles of Mathematical Sciences*, Springer, Berlin, Germany, 2nd edition, 1983.

[100] S. H. Gould, *Variational Methods for Eigenvalue Problems*, University of Toronto Press, Toronto, Ontario, Canada, 1957.

[101] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1967.

[102] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications*, Academic Press, San Diego, Calif, USA, 1995.

[103] P. Hess, "Nonlinear perturbations of linear elliptic and parabolic problems at resonance: existence of multiple solutions," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 5, no. 3, pp. 527–537, 1978.

[104] P. Hess and T. Kato, "On some linear and nonlinear eigenvalue problems with an indefinite weight function," *Communications in Partial Differential Equations*, vol. 5, no. 10, pp. 999–1030, 1980.

[105] J.-B. Hiriart-Urruty, "A short proof of the variational principle for approximate solutions of a minimization problem," *American Mathematical Monthly*, vol. 90, no. 3, pp. 206–207, 1983.

[106] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer, New York, NY, USA, 1975.

[107] E. Hopf, "A remark on linear elliptic differential equations of second order," *Proceedings of the American Mathematical Society*, vol. 3, no. 5, pp. 791–793, 1952.

[108] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, Springer, Berlin, Germany, 1983.

[109] V. Iftimie, *Partial Differential Equations*, Bucharest University Press, Bucharest, Romania, 1980.

[110] I. R. Ionescu, "Viscosity solutions for dynamic problems with slip-rate dependent friction," *Quarterly of Applied Mathematics*, vol. 60, no. 3, pp. 461–476, 2002.

[111] I. R. Ionescu and V. D. Rădulescu, "Nonlinear eigenvalue problems arising in earthquake initiation," *Advances in Differential Equations*, vol. 8, no. 7, pp. 769–786, 2003.

[112] M. N. Islam, *Ultrafast Fiber Switching Devices and Systems*, Cambridge University Press, New York, NY, USA, 1992.

[113] Y. Jabri, *The Mountain Pass Theorem. Variants, Generalizations and Some Applications*, vol. 95 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2003.

[114] D. D. Joseph and T. S. Lundgren, "Quasilinear Dirichlet problems driven by positive sources," *Archive for Rational Mechanics and Analysis*, vol. 49, pp. 241–269, 1973.

[115] J. Jost, *Postmodern Analysis*, Springer, Berlin, Germany, 1998.

[116] T. Kato, "Remarks on holomorphic families of Schrödinger and Dirac operators," in *Differential Equations*, I. Knowles and R. Lewis, Eds., vol. 92 of *North-Holland Mathematics Studies*, pp. 341–352, North-Holland, Amsterdam, The Netherlands, 1984.

[117] J. L. Kazdan and F. W. Warner, "Scalar curvature and conformal deformation of Riemannian structure," *Journal of Differential Geometry*, vol. 10, pp. 113–134, 1975.

[118] J. L. Kazdan and F. W. Warner, "Remarks on some quasilinear elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 28, no. 5, pp. 567–597, 1975.

[119] H. B. Keller and D. S. Cohen, "Some positone problems suggested by nonlinear heat generation," *Journal of Mathematics and Mechanics*, vol. 16, pp. 1361–1376, 1967.

[120] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations*, Springer, Berlin, Germany, 2003.

[121] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1980.

[122] B. Knaster, C. Kuratowski, and S. Mazurkiewicz, "Ein beweis des fixpunktsatzes für *n*-dimensionale simplexe," *Fundamenta Mathematicae*, vol. 14, pp. 132–137, 1929.

[123] A. Kolmogoroff, I. Petrovsky, and N. Piscounoff, "Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem," *Moscow University Mathematics Bulletin*, vol. 1, pp. 1–25, 1937 (French).

[124] S. G. Krantz and H. R. Parks, *The Implicit Function Theorem. History, Theory, and Applications*, Birkhäuser, Boston, Mass, USA, 2002.

[125] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, The Netherlands, 1964.

[126] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, NY, USA, 1964.

[127] A. Kristály, "Infinitely many radial and non-radial solutions for a class of hemivariational inequalities," *The Rocky Mountain Journal of Mathematics*, vol. 35, no. 4, pp. 1173–1190, 2005.

[128] A. Kristály and C. Varga, "A set-valued approach to hemivariational inequalities," *Topological Methods in Nonlinear Analysis*, vol. 24, no. 2, pp. 297–307, 2004.

[129] C. Kuratowski, *Topologie I. Espaces Métrisables, Espaces Complets*, vol. 3 of *Monografje Matematyczne*, Subwencji Funduszu Kultury Narodowej, Warszawa, Poland, 1933.

[130] T. Laetsch, "A note on a paper of Keller and Cohen," *Journal of Mathematics and Mechanics*, vol. 18, pp. 1095–1100, 1969.

[131] E. M. Landesman and A. C. Lazer, "Nonlinear perturbations of linear elliptic boundary value problems at resonance," *Journal of Mathematics and Mechanics*, vol. 19, pp. 609–623, 1970.

[132] G. Lebourg, "Valeur moyenne pour gradient généralisé," *Comptes Rendus de l'Académie des Sciences. Paris*, vol. 281, no. 19, pp. 795–797, 1975.

[133] C. Lefter and V. D. Rădulescu, "Minimization problems and corresponding renormalized energies," *Differential and Integral Equations*, vol. 9, no. 5, pp. 903–917, 1996.

[134] S. Li, "Some aspects of nonlinear operators and critical point theory," in *Functional Analysis in China*, B. Li, Ed., vol. 356 of *Mathematics and Its Applications*, pp. 132–144, Kluwer Academic, Dordrecht, The Netherlands, 1996.

[135] Y. Li and W.-M. Ni, "On conformal scalar curvature equations in $\mathbb{R}^n$," *Duke Mathematical Journal*, vol. 57, no. 3, pp. 895–924, 1988.

[136] T.-C. Lin and J. Wei, "Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$," *Communications in Mathematical Physics*, vol. 255, no. 3, pp. 629–653, 2005.

[137] P.-L. Lions, "The concentration-compactness principle in the calculus of variations. The locally compact case. I," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 1, no. 2, pp. 109–145, 1984.

[138] P.-L. Lions, "The concentration-compactness principle in the calculus of variations. The locally compact case. II," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 1, no. 4, pp. 223–283, 1984.

[139] L. Lusternik, "Topologische Grundlagen der allgemeinen Eigenwerttheorie," *Monatshefte für Mathematik und Physik*, vol. 37, no. 1, pp. 125–130, 1930.

[140] L. Ljusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationnels*, Hermann et Cie, Paris, France, 1934.

[141] A. Marino and D. Scolozzi, "Geodesics with obstacles," *Bollettino della Unione Matemàtica Italiana. B. Serie VI*, vol. 2, no. 1, pp. 1–31, 1983.

[142] J. Mawhin, "Boundary value problems for nonlinear ordinary differential equations: from successive approximations to topology," in *Development of Mathematics 1900–1950*, J.-P. Pier, Ed., pp. 443–477, Birkhäuser, Basel, Switzerland, 1994.

[143] J. Mawhin and M. Willem, "Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations," *Journal of Differential Equations*, vol. 52, no. 2, pp. 264–287, 1984.

[144] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[145] C. R. Menyuk, "Pulse propagation in an elliptically birefringent Kerr medium," *IEEE Journal of Quantum Electronics*, vol. 25, no. 12, pp. 2674–2682, 1989.

[146] M. Mihăilescu and V. D. Rădulescu, "A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids," *Proceedings of the Royal Society of London. Series A*, vol. 462, no. 2073, pp. 2625–2641, 2006.

[147] M. Mihăilescu and V. D. Rădulescu, "Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 416–432, 2007.

[148] M. Mihăilescu and V. D. Rădulescu, "Nonhomogeneous boundary value problems in Orlicz-Sobolev spaces," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 344, no. 1, pp. 15–20, 2007.

[149] M. Mihăilescu and V. D. Rădulescu, "On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent," *Proceedings of the American Mathematical Society*, vol. 135, no. 9, pp. 2929–2937, 2007.

[150] S. Minakshisundaram and Å. Pleijel, "Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds," *Canadian Journal of Mathematics*, vol. 1, pp. 242–256, 1949.

[151] P. Mironescu and V. D. Rădulescu, "A bifurcation problem associated to a convex, asymptotically linear function," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 316, no. 7, pp. 667–672, 1993.

[152] P. Mironescu and V. D. Rădulescu, "Periodic solutions of the equation $-\Delta v = v(1 - |v|^2)$ in $\mathbb{R}$ and $\mathbb{R}^2$," *Houston Journal of Mathematics*, vol. 20, no. 4, pp. 653–669, 1994.

[153] P. Mironescu and V. D. Rădulescu, "A multiplicity theorem for locally Lipschitz periodic functionals," *Journal of Mathematical Analysis and Applications*, vol. 195, no. 3, pp. 621–637, 1995.

[154] P. Mironescu and V. D. Rădulescu, "The study of a bifurcation problem associated to an asymptotically linear function," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 4, pp. 857–875, 1996.

[155] E. Montefusco and V. D. Rădulescu, "Nonlinear eigenvalue problems for quasilinear operators on unbounded domains," *NoDEA: Nonlinear Differential Equations and Applications*, vol. 8, no. 4, pp. 481–497, 2001.

[156] U. Mosco, "Implicit variational problems and quasi-variational inequalities," in *Nonlinear Operators and the Calculus of Variations*, J. P. Gossez, E. J. Lami Dozo, J. Mawhin, and L. Waelbroeck, Eds., vol. 543 of *Lecture Notes in Mathematics*, pp. 83–156, Springer, Berlin, Germany, 1976.

[157] J. Moser, "A new technique for the construction of solutions of nonlinear differential equations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 47, no. 11, pp. 1824–1831, 1961.

[158] D. Motreanu and V. D. Rădulescu, "Existence theorems for some classes of boundary value problems involving the p-Laplacian," *PanAmerican Mathematical Journal*, vol. 7, no. 2, pp. 53–66, 1997.

[159] D. Motreanu and V. D. Rădulescu, "Existence results for inequality problems with lack of convexity," *Numerical Functional Analysis and Optimization*, vol. 21, no. 7-8, pp. 869–884, 2000.

[160] D. Motreanu and V. D. Rădulescu, *Variational and Non-Variational Methods in Nonlinear Analysis and Boundary Value Problems*, vol. 67 of *Nonconvex Optimization and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003.

[161] D. Motreanu and V. D. Rădulescu, "Eigenvalue problems for degenerate nonlinear elliptic equations in anisotropic media," *Boundary Value Problems*, vol. 2005, no. 2, pp. 107–127, 2005.

[162] M. Nagumo, "Über die Differentialgleichung $y'' = f(x, y, y')$," *Proceedings of the Physico-Mathematical Society of Japan*, vol. 19, pp. 861–866, 1937.

[163] M. Naito, "A note on bounded positive entire solutions of semilinear elliptic equations," *Hiroshima Mathematical Journal*, vol. 14, no. 1, pp. 211–214, 1984.

[164] Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, vol. 188 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.

[165] J. Nash, "The imbedding problem for Riemannian manifolds," *Annals of Mathematics. Second Series*, vol. 63, no. 1, pp. 20–63, 1956.

[166] M. Nicolescu, N. Dinculeanu, and S. Marcus, *Mathematical Analysis, Vol. I*, Editura Didactica si Pedagogica, Bucuresti, Romania, 1971.

[167] C. P. Niculescu and V. D. Rădulescu, "A saddle point theorem for non-smooth functionals and problems at resonance," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 21, no. 1, pp. 117–131, 1996.

[168] L. Nirenberg, "Variational and topological methods in nonlinear problems," *Bulletin of the American Mathematical Society. New Series*, vol. 4, no. 3, pp. 267–302, 1981.

[169] L. Nirenberg, "Partial differential equations in the first half of the century," in *Development of Mathematics 1900–1950*, J.-P. Pier, Ed., pp. 479–515, Birkhäuser, Basel, Switzerland, 1994.

[170] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, vol. 6 of *Courant Lecture Notes in Mathematics*, American Mathematical Society, Providence, RI, USA, 2001.

[171] Y.-G. Oh, "Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class (V_a)," *Communications in Partial Differential Equations*, vol. 13, no. 12, pp. 1499–1519, 1988.

[172] M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, "Freak waves in random oceanic sea states," *Physical Review Letters*, vol. 86, no. 25, pp. 5831–5834, 2001.

[173] D. O'Regan and R. Precup, *Theorems of Leray-Schauder Type and Applications*, vol. 3 of *Series in Mathematical Analysis and Applications*, Gordon and Breach Science Publishers, Amsterdam, The Netherlands, 2001.

[174] R. S. Palais, "Lusternik-Schnirelman theory on Banach manifolds," *Topology*, vol. 5, no. 2, pp. 115–132, 1966.

[175] P. D. Panagiotopoulos, *Hemivariational Inequalities: Applications in Mechanics and Engineering*, Springer, Berlin, Germany, 1993.

[176] Å. Pleijel, "On the eigenvalues and eigenfunctions of elastic plates," *Communications on Pure and Applied Mathematics*, vol. 3, no. 1, pp. 1–10, 1950.

[177] R. Precup, *Lectures on Partial Differential Equations*, Presa Universitară Clujeană, Cluj-Napoca, Romania, 2004.

[178] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer, New York, NY, USA, 1984.

[179] P. Pucci and J. Serrin, "Extensions of the mountain pass theorem," *Journal of Functional Analysis*, vol. 59, no. 2, pp. 185–210, 1984.

[180] P. Pucci and J. Serrin, "A mountain pass theorem," *Journal of Differential Equations*, vol. 60, no. 1, pp. 142–149, 1985.

[181] P. Pucci and J. Serrin, "A note on the strong maximum principle for elliptic differential inequalities," *Journal de Mathématiques Pures et Appliquées*, vol. 79, no. 1, pp. 57–71, 2000.

[182] P. Pucci and J. Serrin, "The strong maximum principle revisited," *Journal of Differential Equations*, vol. 196, no. 1, pp. 1–66, 2004.

[183] P. Pucci and J. Serrin, "Erratum: "the strong maximum principle revisited"," *Journal of Differential Equations*, vol. 207, no. 1, pp. 226–227, 2004.

[184] P. H. Rabinowitz, "Some critical point theorems and applications to semilinear elliptic partial differential equations," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, vol. 5, no. 1, pp. 215–223, 1978.

[185] P. H. Rabinowitz, "Free vibrations for a semilinear wave equation," *Communications on Pure and Applied Mathematics*, vol. 31, no. 1, pp. 31–68, 1978.

[186] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[187] P. H. Rabinowitz, "On a class of nonlinear Schrödinger equations," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 43, no. 2, pp. 270–291, 1992.

[188] V. D. Rădulescu, "Sur la théorie de Ljusternik-Schnirelmann en dimension finie," *Matarom, Laboratoire d'Analyse Numérique, Université de Paris VI*, no. 2, pp. 9–17, 1992.

[189] V. D. Rădulescu, "Mountain pass theorems for non-differentiable functions and applications," *Proceedings of Japan Academy. Series A*, vol. 69, no. 6, pp. 193–198, 1993.

[190] V. D. Rădulescu, "Mountain-pass type theorems for nondifferentiable convex functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 39, no. 1, pp. 53–62, 1994.

[191] V. D. Rădulescu, "Locally Lipschitz functionals with the strong Palais-Smale property," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 40, no. 3-4, pp. 355–372, 1995.

[192] V. D. Rădulescu, "A Lusternik-Schnirelmann type theorem for locally Lipschitz functionals with applications to multivalued periodic problems," *Proceedings of Japan Academy. Series A*, vol. 71, no. 7, pp. 164–167, 1995.

[193] V. D. Rădulescu, "Nontrivial solutions for a multivalued problem with strong resonance," *Glasgow Mathematical Journal*, vol. 38, no. 1, pp. 53–59, 1996.

[194] V. D. Rădulescu, "Hemivariational inequalities associated to multivalued problems with strong resonance," in *Nonsmooth/Nonconvex Mechanics: Modeling, Analysis and Numerical Methods*, D. Y. Gao, R. W. Ogden, and G. E. Stavroulakis, Eds., vol. 50 of *Nonconvex Optimization and Its Applications*, pp. 333–348, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[195] V. D. Rădulescu, "Analyse de quelques problèmes aux limites elliptiques non linéaires," Habilitation à Diriger des Recherches, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie Paris VI, 2003.

[196] V. D. Rădulescu, "On a spectral variational problem arising in the study of earthquakes, multiplicity and perturbation from symmetry," in *Control and Boundary Analysis, Proceedings of the 21st IFIP Conference on System Modeling and Optimization, Sophia Antipolis, France, July 2003*, J. Cagnol and J.-P. Zolesio, Eds., vol. 240 of *Lecture Notes in Pure and Applied Mathematics*, pp. 191–202, Marcel Dekker, New York, NY, USA, 2004.

[197] V. D. Rădulescu, "Bifurcation and asymptotics for elliptic problems with singular nonlinearity," in *Studies in Nonlinear Partial Differential Equations: In Honor of Haim Brezis, the 5th European Conference on Elliptic and Parabolic Problems: A Special Trib- ute to the Work of Haim Brezis, Gaeta Italy, May-June 2004*, C. Bandle, H. Berestycki, B. Brighi, et al., Eds., vol. 63 of *Progress in Nonlinear Differential Equations and their Applications*, pp. 349–362, Birkhäuser, Basel, Switzerland, 2005.

[198] V. D. Rădulescu, "Finitely many solutions for a class of boundary value problems with superlinear convex nonlinearity," *Archiv der Mathematik*, vol. 84, no. 6, pp. 538–550, 2005.

[199] V. D. Rădulescu, "Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities," in *Handbook of Differential Equations: Stationary Partial Differential Equations*, M. Chipot, Ed., vol. 4, pp. 483–591, Elsevier, Amsterdam, The Netherlands, 2007.

[200] V. D. Rădulescu, D. Smets, and M. Willem, "Hardy-Sobolev inequalities with remainder terms," *Topological Methods in Nonlinear Analysis*, vol. 20, no. 1, pp. 145–149, 2002.

[201] V. D. Rădulescu and D. Smets, "Critical singular problems on infinite cones," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 6, pp. 1153–1164, 2003.

[202] V. D. Rădulescu and M. Willem, "Linear elliptic systems involving finite Radon measures," *Differential Integral Equations*, vol. 16, no. 2, pp. 221–229, 2003.

[203] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations*, vol. 13 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 2nd edition, 2004.

[204] W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, NY, USA, 1973.

[205] D. H. Sattinger, *Topics in Stability and Bifurcation Theory*, vol. 309 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.

[206] H. H. Schaefer, *Topological Vector Spaces*, Mac Millan Series in Advances Mathematics and Theoretical Physics, The Macmillan, New York, NY, USA, 1966.

[207] M. Schechter, "Nonlinear elliptic boundary value problems at strong resonance," *American Journal of Mathematics*, vol. 112, no. 3, pp. 439–460, 1990.

[208] M. Schechter, *Linking Methods in Critical Point Theory*, Birkhäuser, Boston, Mass, USA, 1999.

[209] G. Scorza Dragoni, "Il problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenziale del secondo ordine," *Giornale di Matematiche di Battaglini*, vol. 69, pp. 77–112, 1931.

[210] R. P. Sperb, *Maximum Principles and Their Applications*, vol. 157 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1981.

[211] G. Stampacchia, *Équations Elliptiques du Second Ordre à Coefficients Discontinus*, Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, QC, Canada, 1966.

[212] M. Struwe, "Quasilinear elliptic eigenvalue problems," *Commentarii Mathematici Helvetici*, vol. 58, no. 3, pp. 509–527, 1983.

[213] M. Struwe, "Large H-surfaces via the mountain-pass-lemma," *Mathematische Annalen*, vol. 270, no. 3, pp. 441–459, 1985.

[214] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, Springer, Berlin, Germany, 3rd edition, 2000.

[215] F. Sullivan, "A characterization of complete metric spaces," *Proceedings of the American Mathematical Society*, vol. 83, no. 2, pp. 345–346, 1981.

[216] A. Szulkin, "Critical point theory of Ljusternik-Schnirelmann type and applications to partial differential equations," in *Minimax Results of Lusternik-Schnirelman Type and Applications*, vol. 107 of *Séminaire de Mathématiques Supérieures*, pp. 35–96, Les Presses de l'Université de Montréal, Montreal, QC, Canada, 1989.

[217] A. Szulkin and M. Willem, "Eigenvalue problems with indefinite weight," *Studia Mathematica*, vol. 135, no. 2, pp. 191–201, 1999.

[218] K. Thews, "Nontrivial solutions of elliptic equations at resonance," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 85, no. 1-2, pp. 119–129, 1980.

[219] K. Tintarev, "A semilinear elliptic problem on unbounded domains with reverse penalty," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 7, pp. 1496–1502, 2006.

[220] K. Tintarev and K.-H. Fieseler, *Concentration Compactness: Functional-Analytic Grounds and Applications*, Imperial College Press, London, UK, 2007.

[221] J. L. Vázquez, "A strong maximum principle for some quasilinear elliptic equations," *Applied Mathematics and Optimization*, vol. 12, no. 3, pp. 191–202, 1984.

[222] M. Willem, *Minimax Theorems*, vol. 24 of *Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, Mass, USA, 1996.

[223] M. Willem, *Analyse Fonctionnelle Élémentaire*, Cassini, Paris, France, 2003.

[224] V. E. Zakharov, "Collapse and self-focusing of Langmuir waves," in *Handbook of Plasma Physics*, M. N. Rosenbluth, R. Z. Sagdeev, A. A. Galeev, and R. N. Sudan, Eds., vol. 2, pp. 81–121, Elsevier, Amsterdam, The Netherlands, 1984.

[225] M. Zworski, "Resonances in physics and geometry," *Notices of the American Mathematical Society*, vol. 46, no. 3, pp. 319–328, 1999.

Index